CHAPTER 3
EQUILIBRIUM

CHAPTER OUTLINE

Bialobrzeski/laif/Redux Pictures

Article 3/1 Introduction

- When a body is in equilibrium, the resultant of all forces acting on it is zero. Thus, the resultant force R and the resultant couple M are both zero, and we have the equilibrium equations
- Equilibrium Conditions (Eq. 3/1)

Force Balance: $\Sigma \mathbf{F} = \mathbf{0}$

Moment Balance: $\Sigma \mathbf{M} = \mathbf{0}$

- These requirements are both necessary and sufficient conditions for equilibrium.

SECTION A Equilibrium in Two Dimensions

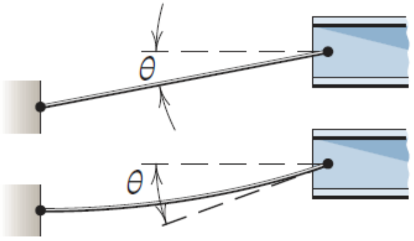
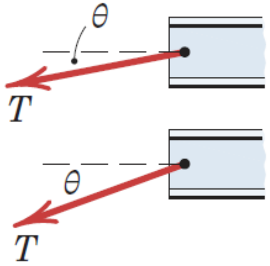
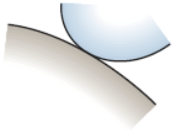
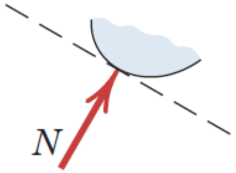
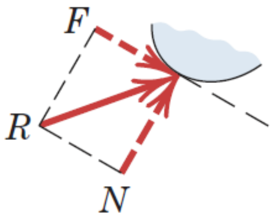
Article 3/2 System Isolation and the Free-Body Diagram

- Before we apply equilibrium equations, we must define unambiguously the particular body or mechanical system to be analyzed and represent clearly and completely all forces acting on the body. Omission of a force which acts on the body in question, or inclusion of a force which does not act on the body, will give erroneous results.
- A *mechanical system* is defined as **a body** or **group of bodies** which can be conceptually isolated from all other bodies. A system may be a single body or a combination of connected bodies. The bodies may be rigid or nonrigid.

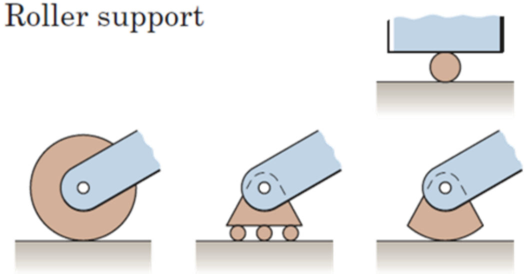
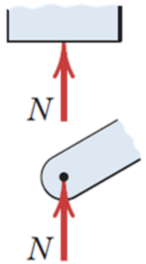
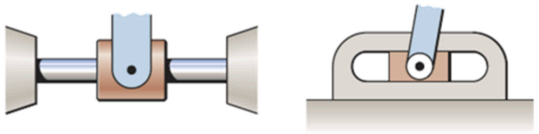
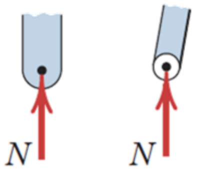
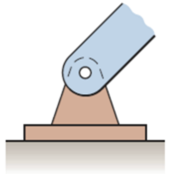
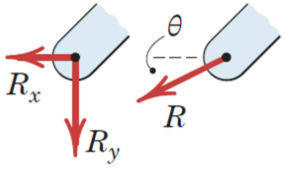
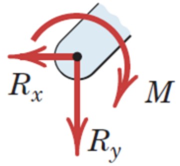
Article 3/2 System Isolation and the Free-Body Diagram

- Once we decide which body or combination of bodies to analyze, we then treat this body or combination as a single body *isolated* from all surrounding bodies.
- This isolation is accomplished by means of ***the free-body diagram***, which is a diagrammatic representation of the isolated system treated as a single body.
- The free-body diagram is the most important single step in the solution of problems in mechanics.

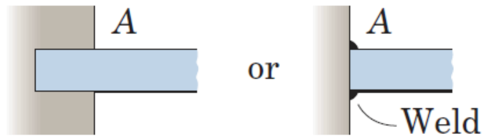
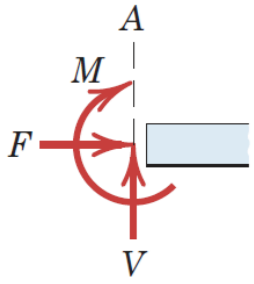
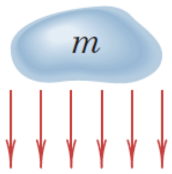
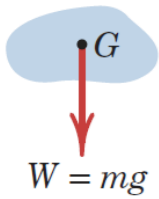
Article 3/2 – Modeling the Action of Forces (1 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to Be Isolated
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component *F* (frictional force) as well as a normal component *N* of the resultant contact force *R*.

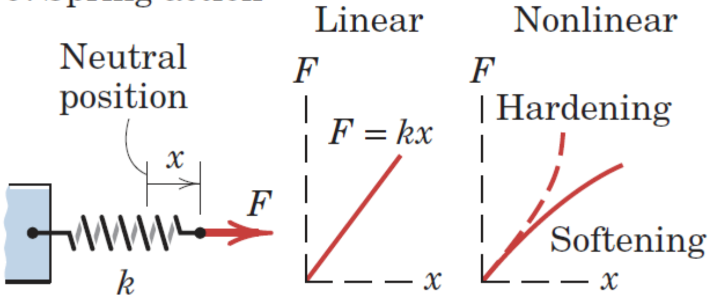
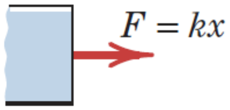
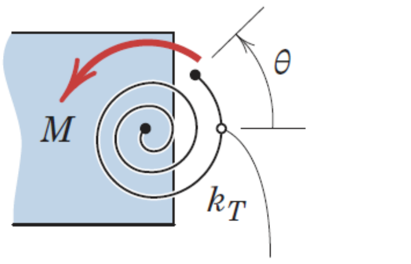
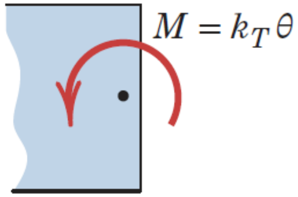
Article 3/2 – Modeling the Action of Forces (2 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	Action on Body to Be Isolated	
4. Roller support 		Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 		Collar or slider free to move along smooth guides; can support force normal to guide only.
6. Pin connection 	Pin free to turn  Pin not free to turn 	A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.

Article 3/2 – Modeling the Action of Forces (3 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to Be Isolated
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center of gravity G.

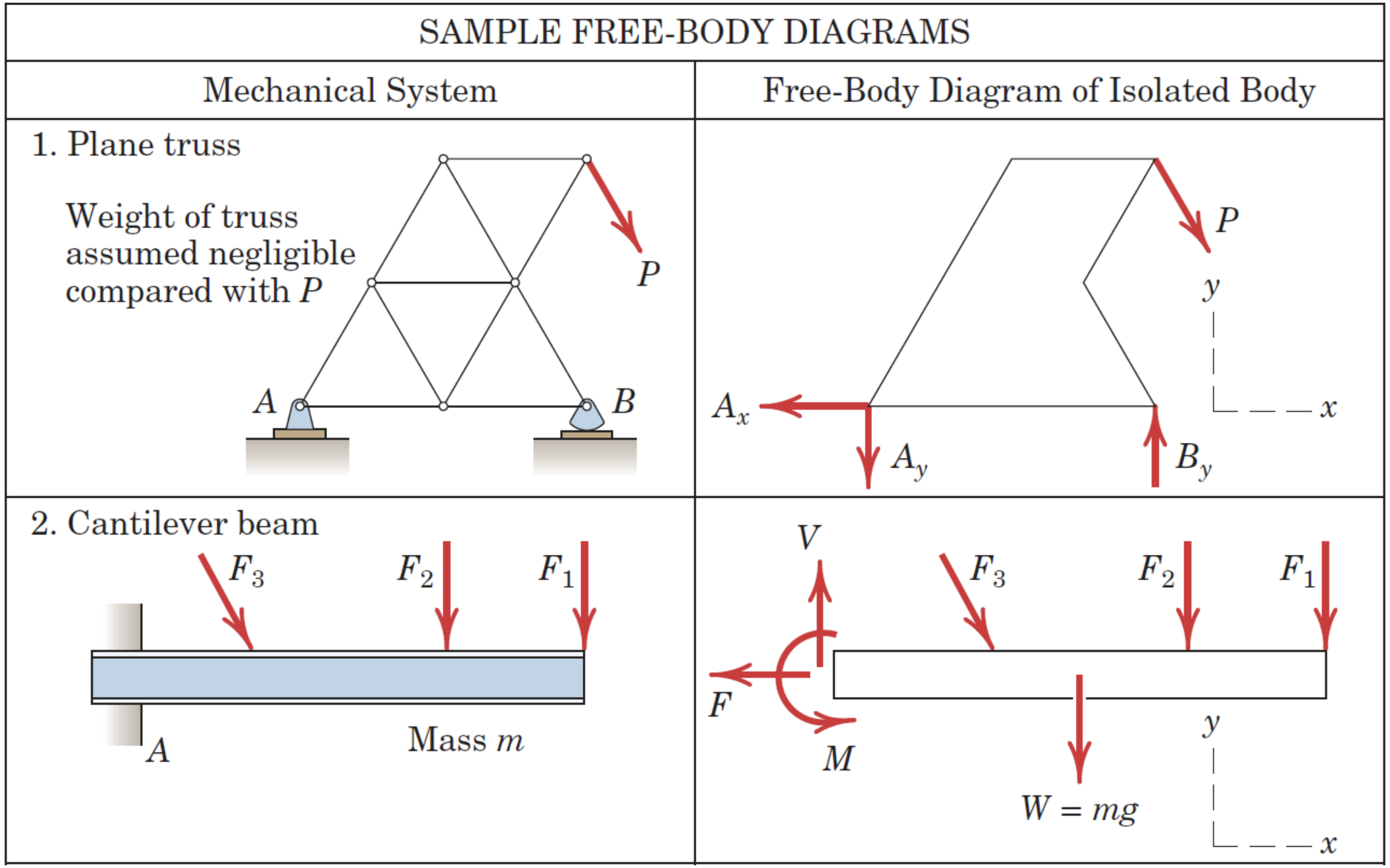
Article 3/2 – Modeling the Action of Forces (4 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to Be Isolated
9. Spring action 	 Spring force is tensile if the spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.
10. Torsional spring action 	 For a linear torsional spring, the applied moment M is proportional to the angular deflection θ from the neutral position. The stiffness k_T is the moment required to deform the spring one radian.

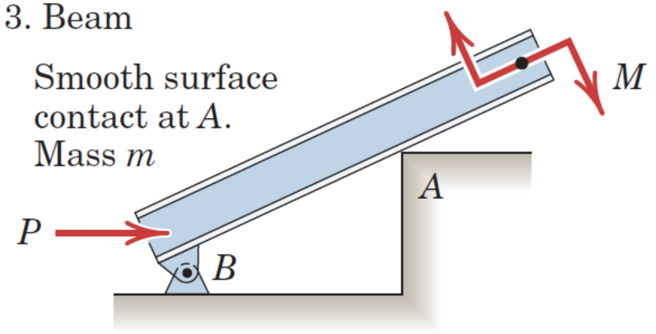
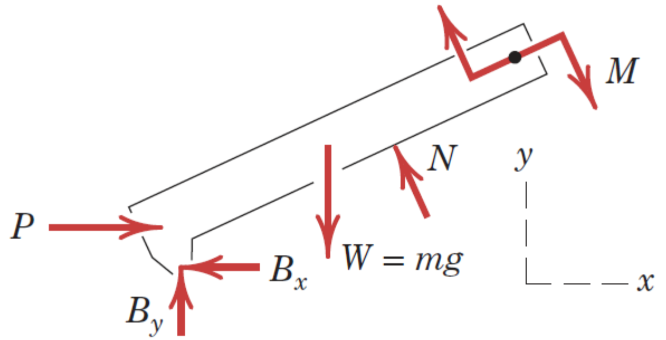
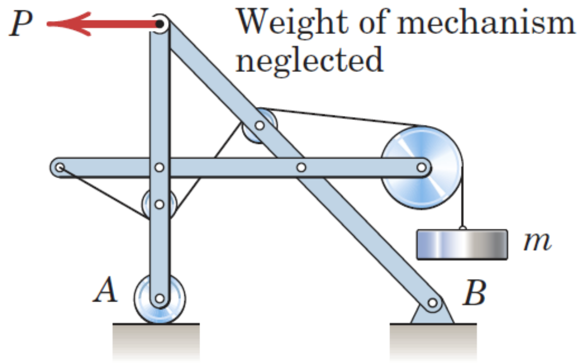
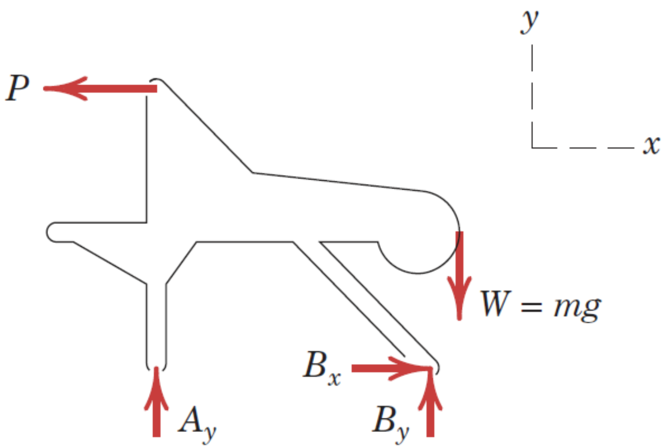
Article 3/2 – Construction of Free-Body Diagrams

- Step 1: Decide which system to isolate.
- Step 2: Isolate the system by drawing a diagram which represents its complete external boundary.
- Step 3: Identify all forces which act on the isolated system as applied by the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system.
- Step 4: Show the choice of coordinate axes directly on the diagram.

Article 3/2 – Examples of Free-Body Diagrams (1 of 2)

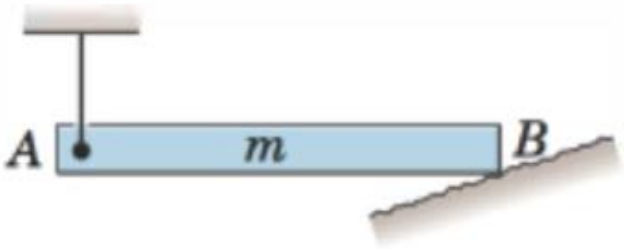


Article 3/2 – Examples of Free-Body Diagrams (2 of 2)

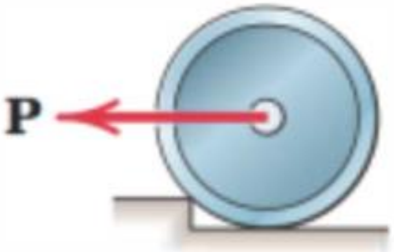
SAMPLE FREE-BODY DIAGRAMS	
Mechanical System	Free-Body Diagram of Isolated Body
3. Beam Smooth surface contact at A. Mass m 	
4. Rigid system of interconnected bodies analyzed as a single unit Weight of mechanism neglected 	

Examples

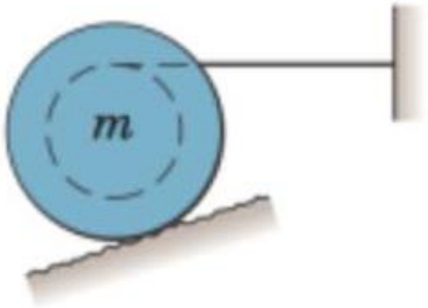
1. Uniform horizontal bar of mass m suspended by vertical cable at A and supported by rough inclined surface at B.



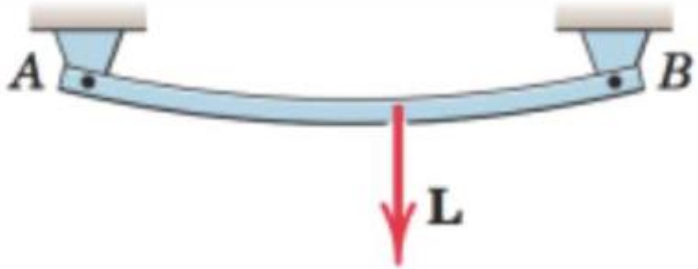
2. Wheel of mass m on verge of being rolled over curb by pull **P**.



5. Uniform grooved wheel of mass m supported by a rough surface and by action of horizontal cable.

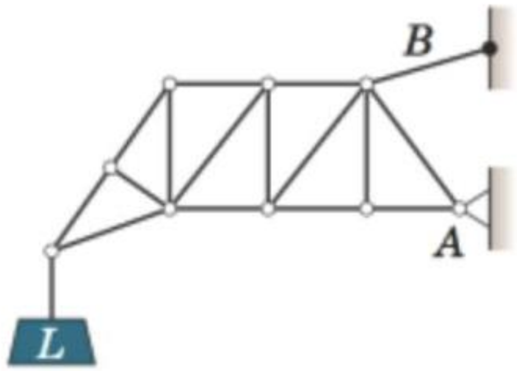


6. Bar, initially horizontal but deflected under load **L**. Pinned to rigid support at each end.

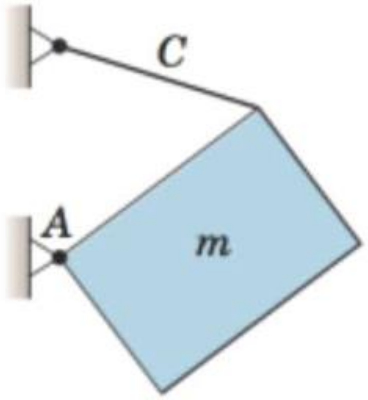


Examples

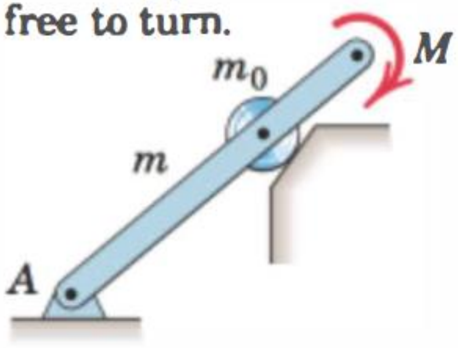
3. Loaded truss supported by pin joint at *A* and by cable at *B*.



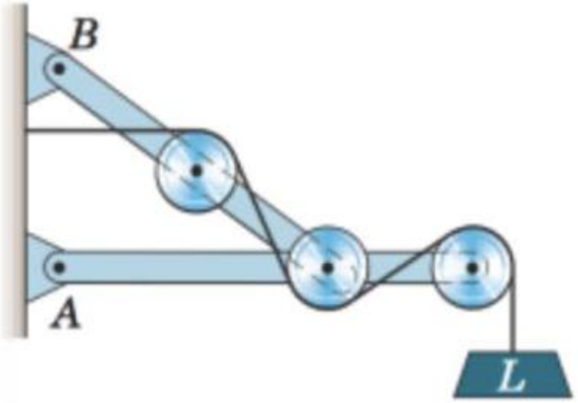
7. Uniform heavy plate of mass *m* supported in vertical plane by cable *C* and hinge *A*.



4. Uniform bar of mass *m* and roller of mass m_0 taken together. Subjected to couple *M* and supported as shown. Roller is free to turn.



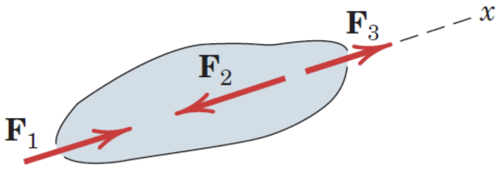
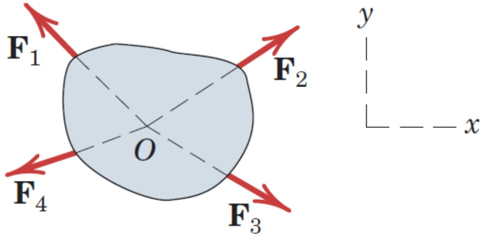
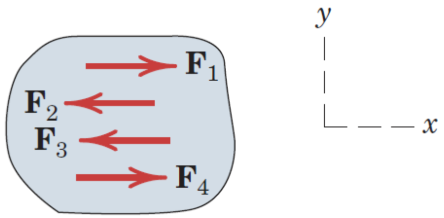
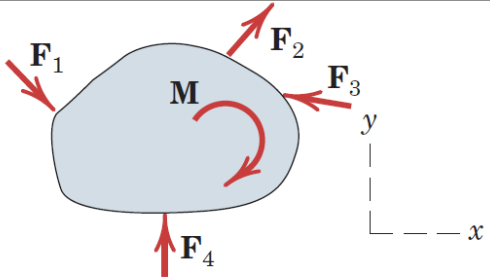
8. Entire frame, pulleys, and contacting cable to be isolated as a single unit.



Article 3/3 Equilibrium Conditions

- Scalar Format (Eq. 3/2)
  - $\Sigma F_x = 0$
  - $\Sigma F_y = 0$
  - $\Sigma M_O = 0$

Article 3/3 – Categories of Equilibrium

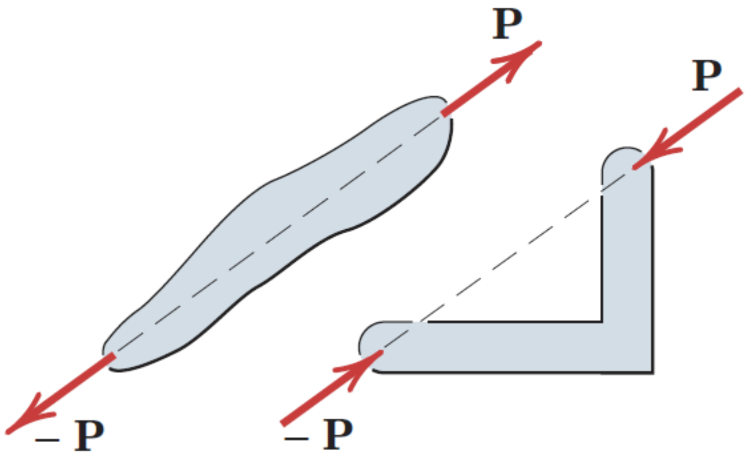
CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

Article 3/3 – Two-Force Members

- Definition

  2-Force Member (2FM) occurs when a body is in equilibrium under the action of two forces only.

- Forces are…
  - Equal
  - Opposite
  - Collinear
  - Independent of Object Shape



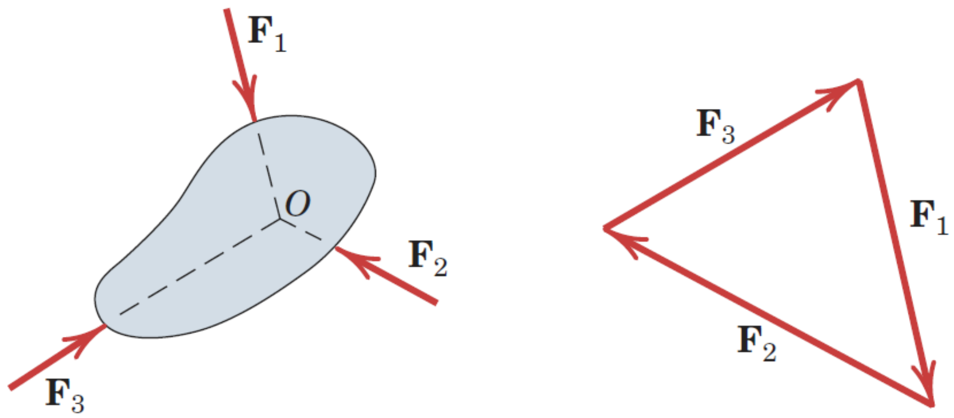
Article 3/3 – Three-Force Members

- Definition

  Occurs when a body is in equilibrium under the action of three forces only.

- Forces are Concurrent

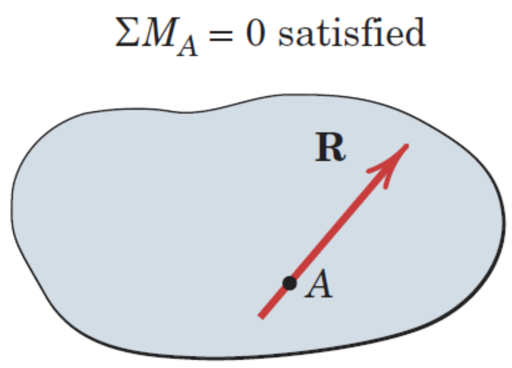
- Case of Parallel Forces



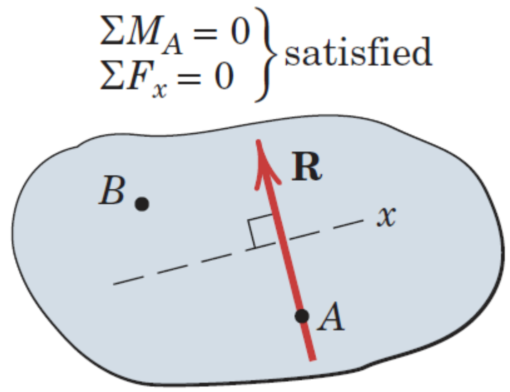
(a) Three-force member (b) Closed polygon satisfies $\Sigma\mathbf{F} = \mathbf{0}$

Article 3/3 – Alternative Equilibrium Equations (1 of 2)

- Case 1: Use of Two Moment Equations
  - Case (*a*)
    - Resultant Intersects Point *A*
  - Case (*b*)
    - Resultant Intersects Point *A*
    - Resultant is Perpendicular to *x*-axis
  - Equilibrium Conditions
    - $\Sigma F_x = 0$
    - $\Sigma M_A = 0$
    - $\Sigma M_B = 0$



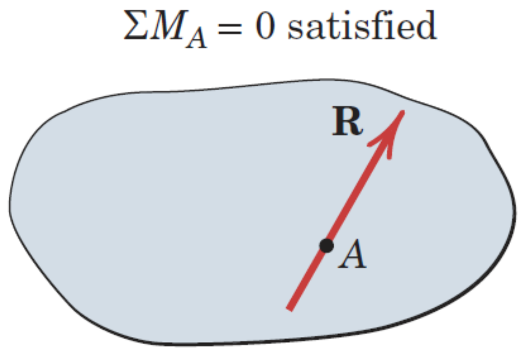
(*a*)



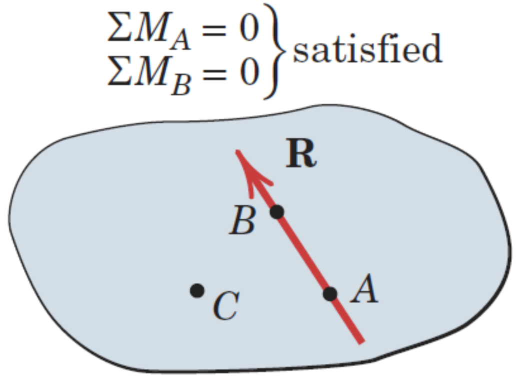
(*b*)

Article 3/3 – Alternative Equilibrium Equations (2 of 2)

- Case 2: Use of Three Moment Equations
  - Case (*c*)
    - Resultant Intersects Point *A*
  - Case (*d*)
    - Resultant Intersects Point *A*
    - Resultant Intersects Point *B*
  - Equilibrium Conditions
    - $\Sigma M_A = 0$
    - $\Sigma M_B = 0$
    - $\Sigma M_C = 0$



(*c*)



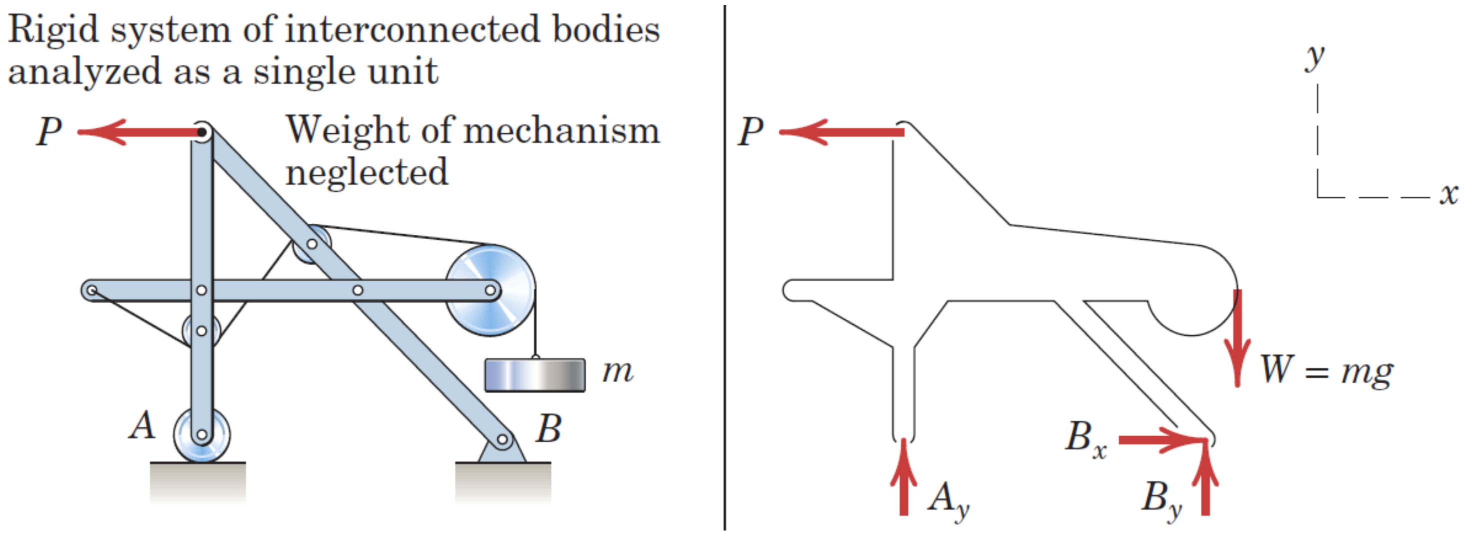
(*d*)

Article 3/3 – Constraints and Statical Determinacy

- Important Terms
  - Constraint
  - Statically Determinant
  - Statically Indeterminant
  - Redundancy
  - Degree of Statical Indeterminacy

Article 3/3 – Illustration of Determinacy

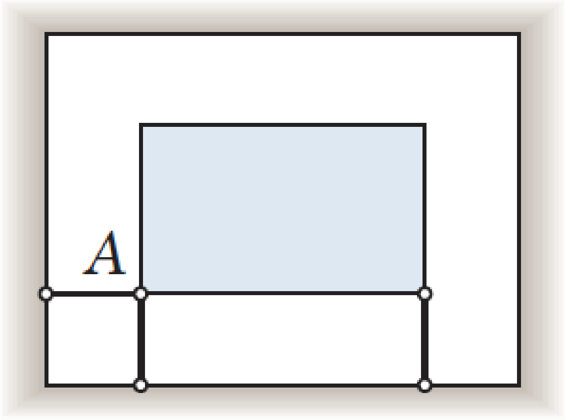
- Statically Determinant System



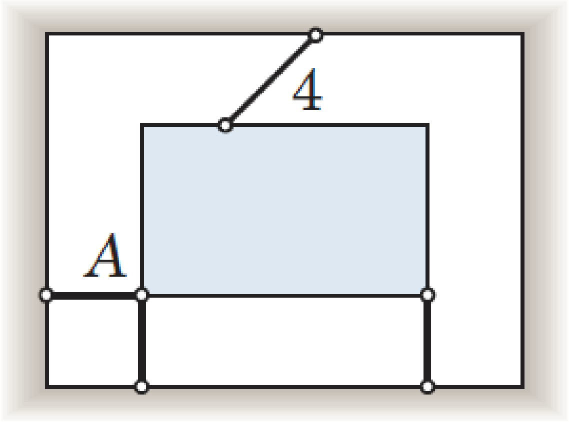
- Statically Indeterminant System – Replace Roller *A* with a Pin

Article 3/3 – Adequacy of Constraints (1 of 2)

- Complete Fixity (Adequate Constraints)

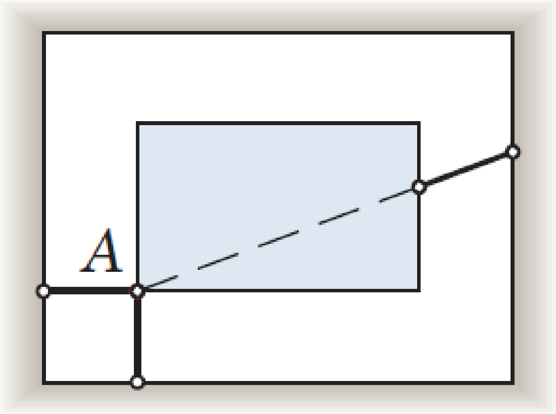


- Excessive Fixity (Redundant Constraints)

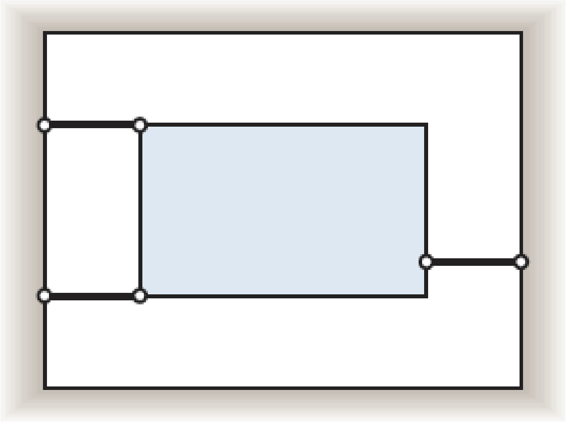


Article 3/3 – Adequacy of Constraints (2 of 2)

- Incomplete Fixity (Partial Constraints)
  - Concurrent Reactions



- Incomplete Fixity (Partial Constraints)
  - Parallel Reactions



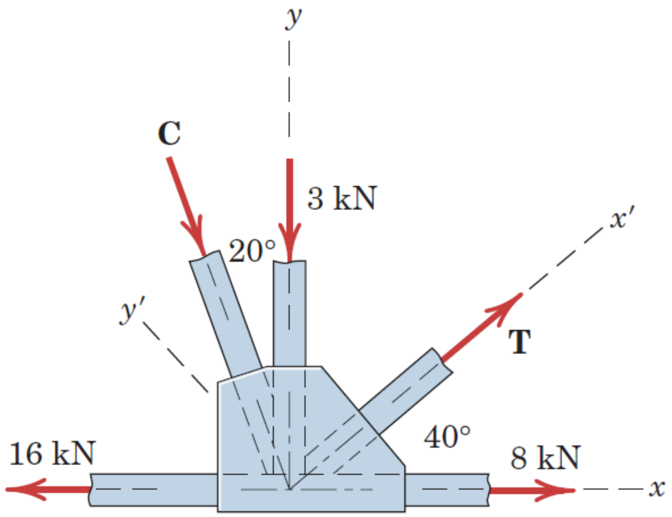
Article 3/3 – Approach to Solving Problems

- Identify clearly the quantities which are known and unknown.
- Make an unambiguous choice of the body (or system) to be isolated and draw its complete free-body diagram.
- Choose a convenient set of reference axes.
- Identify and state the applicable force and moment equations which govern the equilibrium conditions of the problem.
- Match the number of independent equations with the number of unknowns in the problem.
- Carry out the solution and check the results.

Article 3/3 – Sample Problem 3/1 (1 of 4)

- Problem Statement

  Determine the magnitudes of the forces **C** and **T**, which, along with the other three forces shown, act on the bridge-truss joint.



Article 3/3 – Sample Problem 3/1 (2 of 4)

- Solution I (Scalar Algebra) with *x*-*y* axes

$[\Sigma F_x = 0]$ $\quad 8 + T \cos 40° + C \sin 20° - 16 = 0$

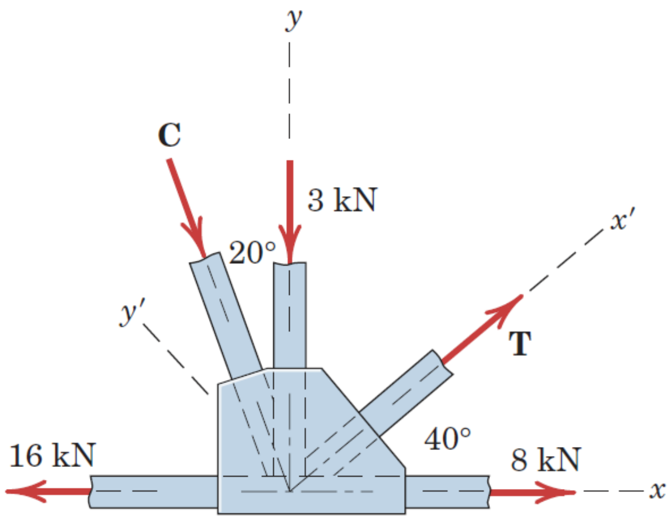
$$0.766T + 0.342C = 8 \qquad (a)$$

$[\Sigma F_y = 0]$ $\quad T \sin 40° - C \cos 20° - 3 = 0$

$$0.643T - 0.940C = 3 \qquad (b)$$

Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN} \qquad \textit{Ans.}$$



Article 3/3 – Sample Problem 3/1 (3 of 4)

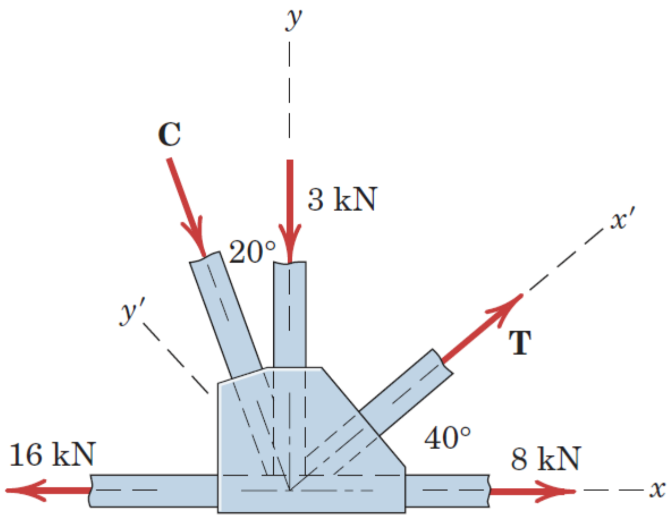
- Solution II (Scalar Algebra) with *x′*-*y′* axes

$[\Sigma F_{y'} = 0]$ $\quad -C \cos 20° - 3 \cos 40° - 8 \sin 40° + 16 \sin 40° = 0$

$C = 3.03 \text{ kN}$ *Ans.*

$[\Sigma F_{x'} = 0]$ $\quad T + 8 \cos 40° - 16 \cos 40° - 3 \sin 40° - 3.03 \sin 20° = 0$

$T = 9.09 \text{ kN}$ *Ans.*



Article 3/3 – Sample Problem 3/1 (4 of 4)

- Solution III (Vector Algebra) with *x*-*y* axes

$[\Sigma \mathbf{F} = \mathbf{0}]$ $\quad 8\mathbf{i} + (T \cos 40°)\mathbf{i} + (T \sin 40°)\mathbf{j} - 3\mathbf{j} + (C \sin 20°)\mathbf{i}$

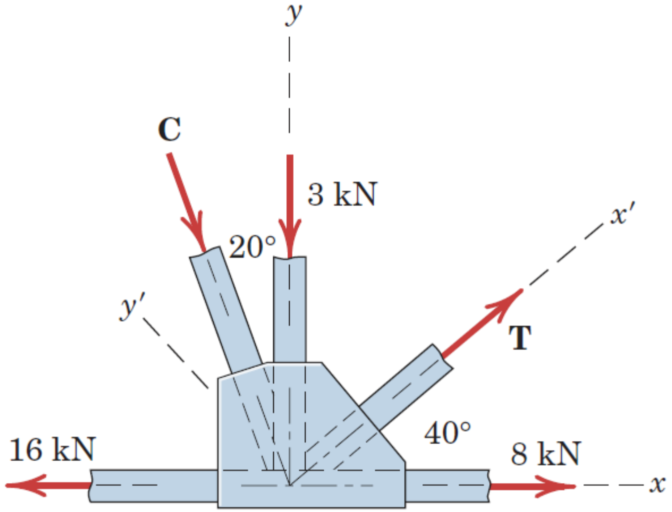
$$- (C \cos 20°)\mathbf{j} - 16\mathbf{i} = \mathbf{0}$$

Equating the coefficients of the **i**- and **j**-terms to zero gives

$$8 + T \cos 40° + C \sin 20° - 16 = 0$$

$$T \sin 40° - 3 - C \cos 20° = 0$$

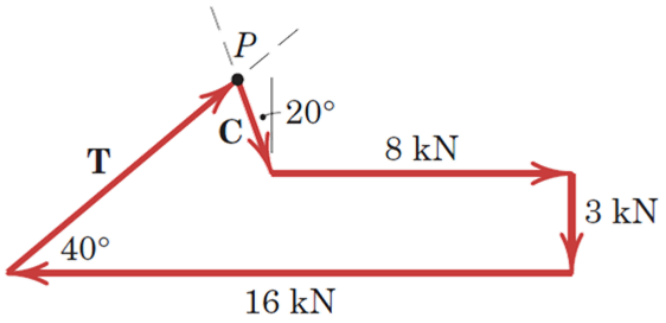
which are the same, of course, as Eqs. (a) and (b), which we solved above.

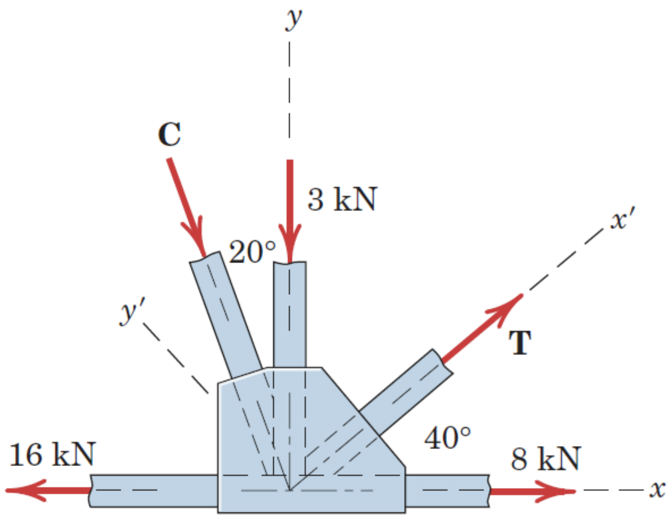


Article 3/3 – Sample Problem 3/1 (4 of 4)

- Solution IV (Geometric)

A graphical solution is easily obtained. The known vectors are laid off head-to-tail to some convenient scale, and the directions of **T** and **C** are then drawn to close the polygon. ③ The resulting intersection at point P completes the solution, thus enabling us to measure the magnitudes of **T** and **C** directly from the drawing to whatever degree of accuracy we incorporate in the construction.



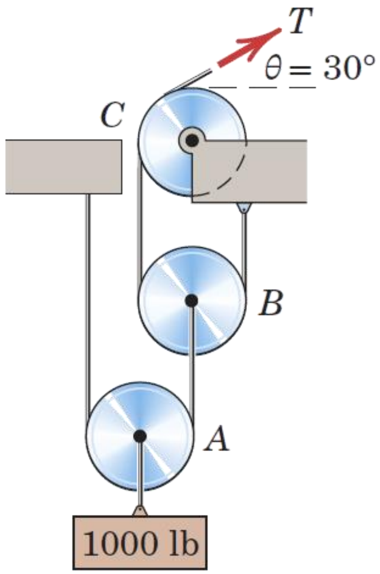


③ The known vectors may be added in any order desired, but they must be added before the unknown vectors.

Article 3/3 – Sample Problem 3/2 (1 of 3)

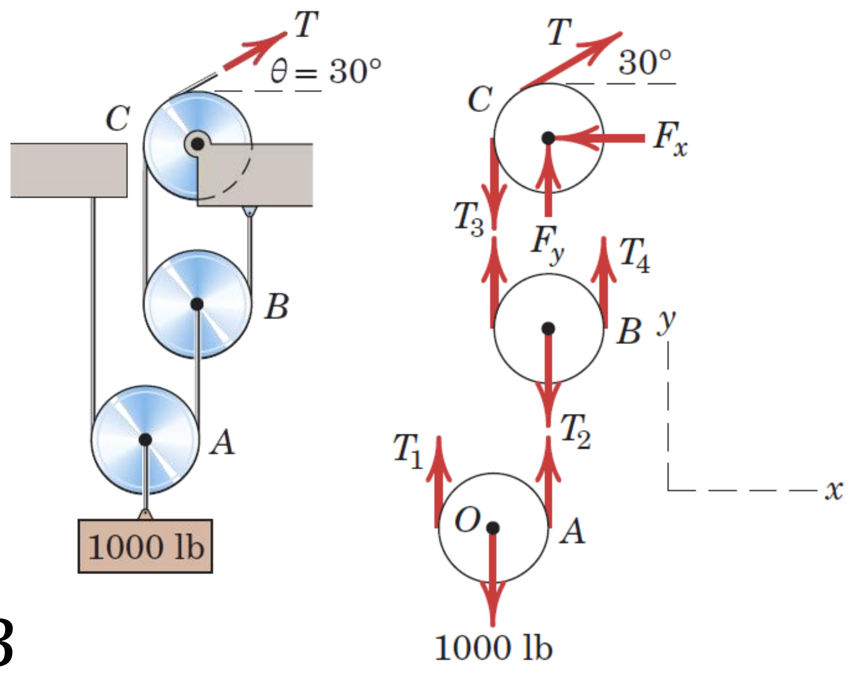
- Problem Statement

Calculate the tension *T* in the cable which supports the 1000-lb load with the pulley arrangement shown. Each pulley is free to rotate about its b earing, and the weights of all parts are small compared with the load. Find the magnitude of the total force on the bearing of pulley *C*.



Article 3/3 – Sample Problem 3/2 (2 of 3)

- Free-Body Diagrams



- Equilibrium Conditions for Pulleys *A* and *B*

$[\Sigma M_O = 0]$ $\quad T_1 r - T_2 r = 0 \quad T_1 = T_2$ ①

$[\Sigma F_y = 0]$ $\quad T_1 + T_2 - 1000 = 0 \quad 2T_1 = 1000 \quad T_1 = T_2 = 500 \text{ lb}$

From the example of pulley *A* we may write the equilibrium of forces on pulley *B* by inspection as

$$T_3 = T_4 = T_2/2 = 250 \text{ lb}$$

① Clearly the radius *r* does not influence the results. Once we have analyzed a simple pulley, the results should be perfectly clear by inspection.

Article 3/3 – Sample Problem 3/2 (3 of 3)

- Equilibrium Conditions for Pulley *C*

For pulley C the angle $\theta = 30°$ in no way affects the moment of T about the center of the pulley, so that moment equilibrium requires

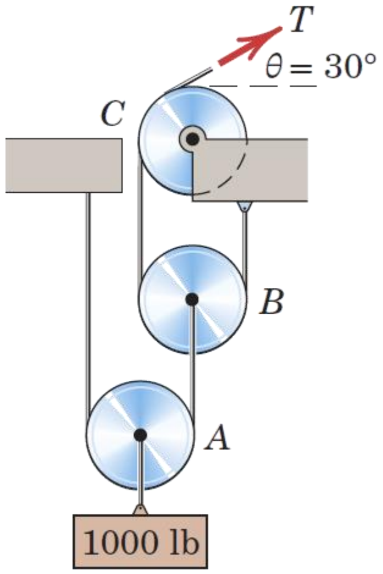
$$T = T_3 \qquad \text{or} \qquad T = 250 \text{ lb} \qquad \textit{Ans.}$$

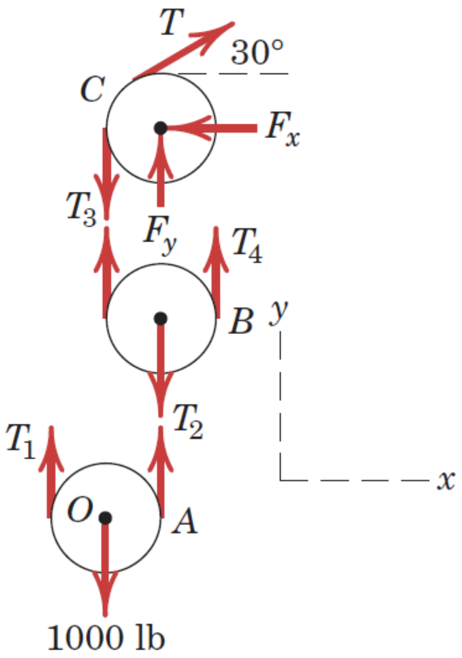
Equilibrium of the pulley in the x- and y-directions requires

$$[\Sigma F_x = 0] \qquad 250 \cos 30° - F_x = 0 \qquad F_x = 217 \text{ lb}$$

$$[\Sigma F_y = 0] \qquad F_y + 250 \sin 30° - 250 = 0 \qquad F_y = 125 \text{ lb}$$

$$\left[F = \sqrt{F_x^{\,2} + F_y^{\,2}}\right] \qquad F = \sqrt{(217)^2 + (125)^2} = 250 \text{ lb} \qquad \textit{Ans.}$$

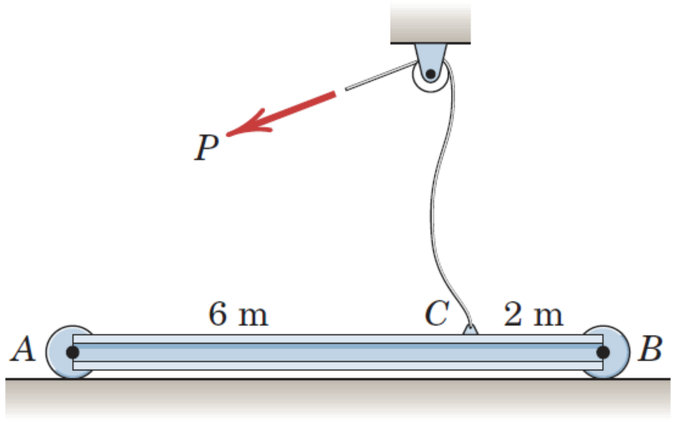




Article 3/3 – Sample Problem 3/3 (1 of 2)

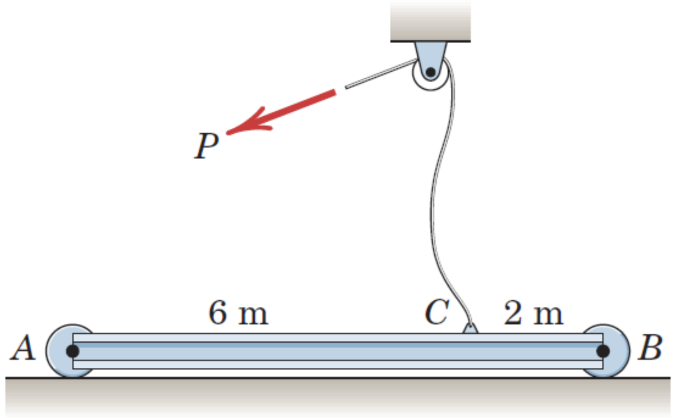
- Problem Statement

The uniform 100-kg I-beam is supported initially by its end rollers on the horizontal surface at *A* and *B*. By means of the cable at *C*, it is desired to elevate end *B* to a position 3 m above end *A*. Determine the required tension *P*, the reaction at *A*, and the angle θ made by the beam with the horizontal in the elevated position.



Article 3/3 – Sample Problem 3/3 (2 of 2)

- Free-Body Diagram



- Equilibrium Conditions

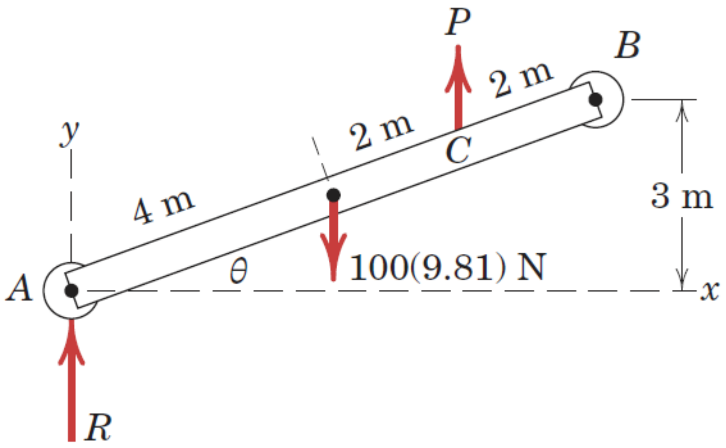
$[\Sigma M_A = 0]$ $\quad P(6 \cos \theta) - 981(4 \cos \theta) = 0 \quad P = 654 \text{ N}$ ① *Ans.*

Equilibrium of vertical forces requires

$[\Sigma F_y = 0]$ $\quad 654 + R - 981 = 0 \quad R = 327 \text{ N}$ *Ans.*

The angle θ depends only on the specified geometry and is

$$\sin \theta = 3/8 \qquad \theta = 22.0°$$ *Ans.*

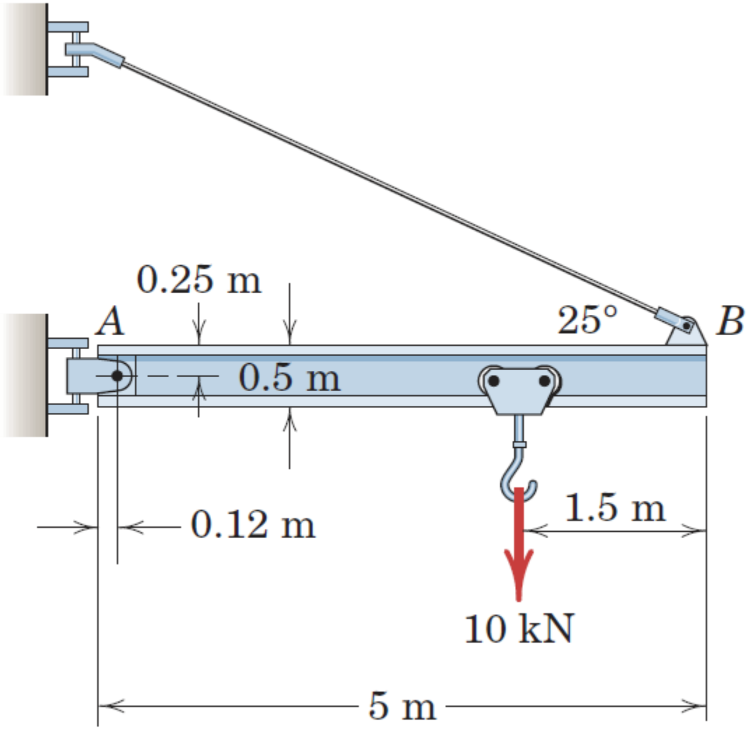


① Clearly the equilibrium of this parallel force system is independent of θ.

Article 3/3 – Sample Problem 3/4 (1 of 3)

- Problem Statement

Determine the magnitude *T* of the tension in the supporting cable and the magnitude of the force on the pin at *A* for the jib crane shown. The beam *AB* is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.



Article 3/3 – Sample Problem 3/4 (2 of 3)

- Free-Body Diagram

- Equilibrium Conditions

$[\Sigma M_A = 0]$ $\quad (T \cos 25°)0.25 + (T \sin 25°)(5 - 0.12)$

$\quad - 10(5 - 1.5 - 0.12) - 4.66(2.5 - 0.12) = 0$ ②

from which $\quad T = 19.61 \text{ kN}$ $\quad$ *Ans.*

Equating the sums of forces in the x- and y-directions to zero gives

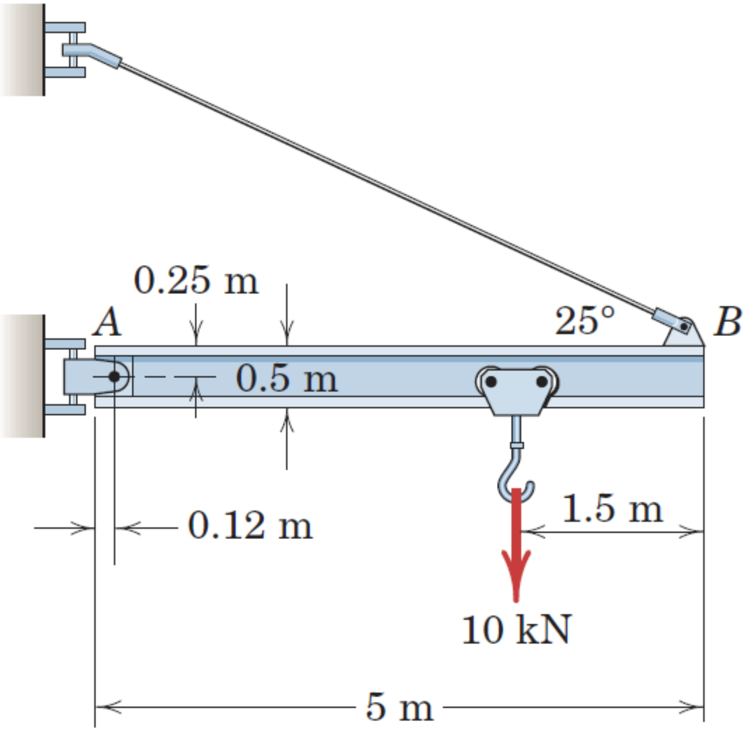
$[\Sigma F_x = 0]$ $\quad A_x - 19.61 \cos 25° = 0$ $\quad A_x = 17.77 \text{ kN}$

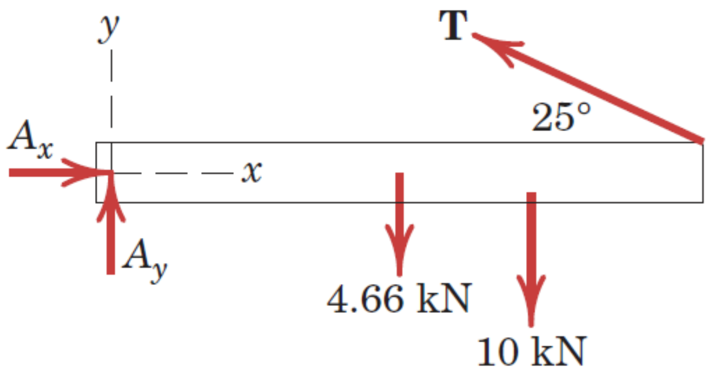
$[\Sigma F_y = 0]$ $\quad A_y + 19.61 \sin 25° - 4.66 - 10 = 0$ $\quad A_y = 6.37 \text{ kN}$

$[A = \sqrt{A_x^2 + A_y^2}]$ $\quad A = \sqrt{(17.77)^2 + (6.37)^2} = 18.88 \text{ kN}$ ③ $\quad$ *Ans.*

② The calculation of moments in two-dimensional problems is generally handled more simply by scalar algebra than by the vector cross product **r** × **F**. In three dimensions, as we will see later, the reverse is often the case.

③ The direction of the force at A could be easily calculated if desired. However, in designing the pin A or in checking its strength, it is only the magnitude of the force that matters.





Article 3/3 – Sample Problem 3/4 (3 of 3)

- Graphical Solution