Deflection of Beams

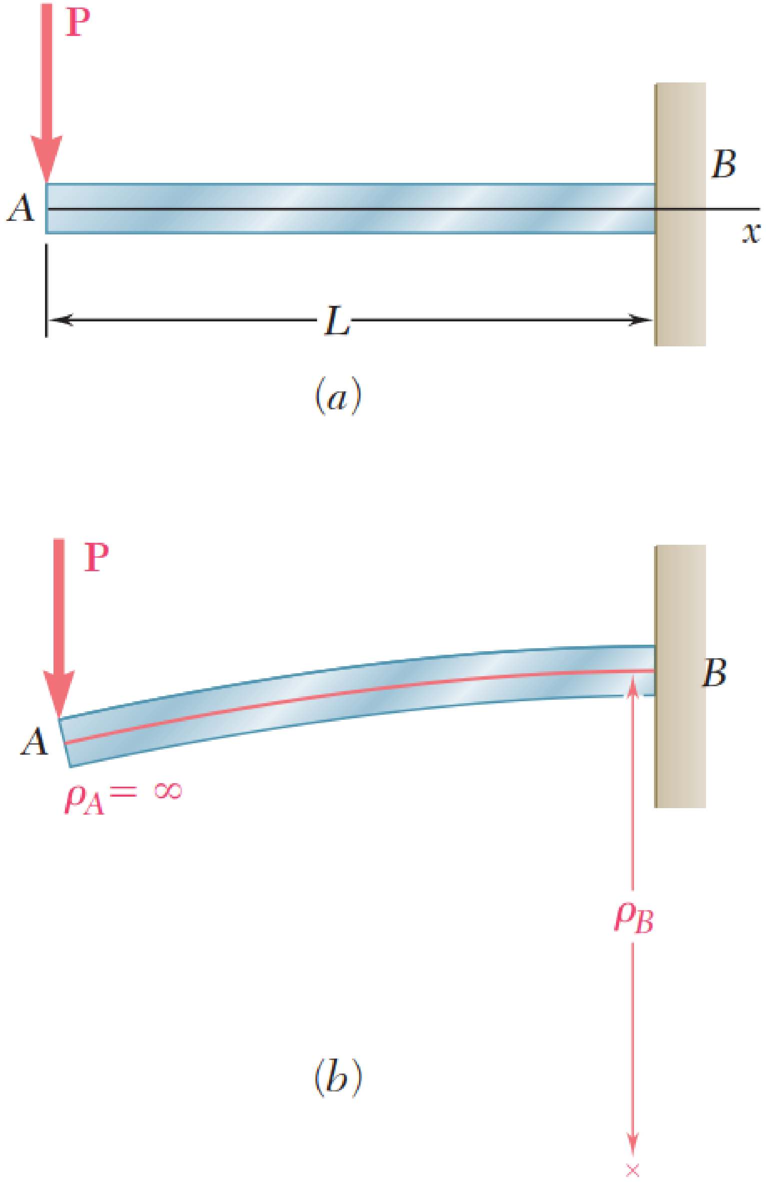
9

Objectives

- ✓ Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:
  - ➢ The integration method
  - ➢ The use of discontinuity functions
  - ➢ The method of superposition
- ✓ Determine the same, using a semi-graphical technique, called the moment-area method.

APPLICATIONS

DEFORMATION UNDER TRANSVERSE LOADING



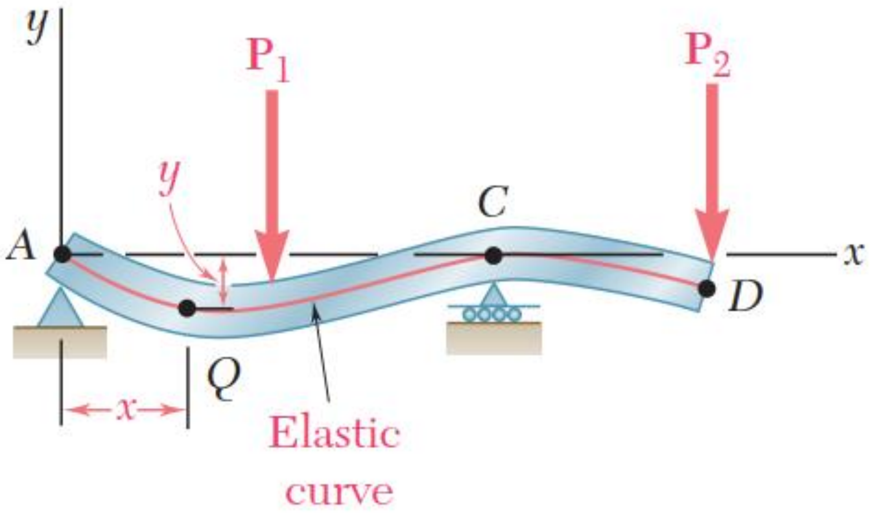
The curvature of the neutral surface to the bending moment in a beam in pure bending is

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

However, both the bending moment and the curvature of the neutral surface vary from section to section. Denoting by *x* the distance of the section from the left end of the beam,

The shape of the deformed beam is obtained from the information about its curvature. However, the analysis and design of a beam usually requires more precise information on the *deflection* and the *slope* at various points. Of particular importance is the maximum deflection of the beam.

Equation of The Elastic Curve



The curvature of a plane curve at a point *Q*(*x*,*y*) is

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

For the elastic curve of a beam, however, the slope *dy*/*dx* is very small, and its square is negligible compared to unity. Therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

This equation is a second-order linear differential equation; it is the governing differential equation **for the elastic curve.**

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

Equation of The Elastic Curve

The product *EI* is called the *flexural rigidity.* For a prismatic beam, the flexural rigidity constant.

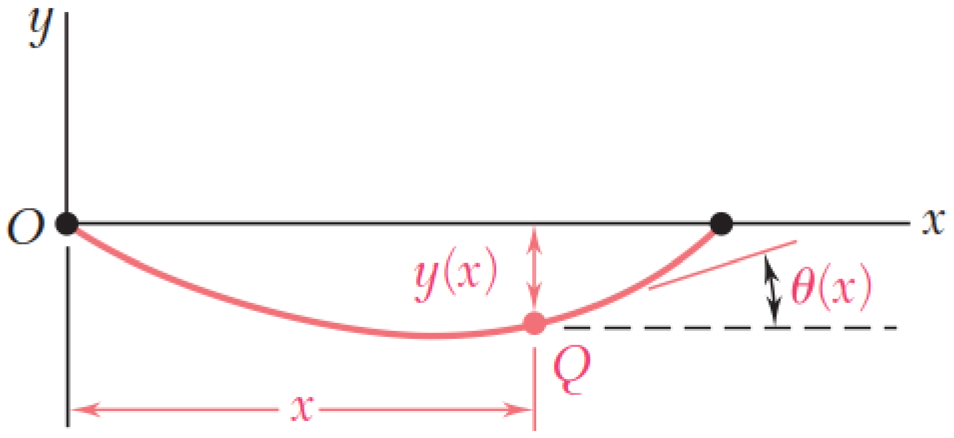


Fig. 9.7 Slope $\theta(x)$ of tangent to the elastic curve.

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1$$

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x) \longrightarrow \quad EI\,\theta(x) = \int_0^x M(x)\,dx + C_1$$

Slope Equation

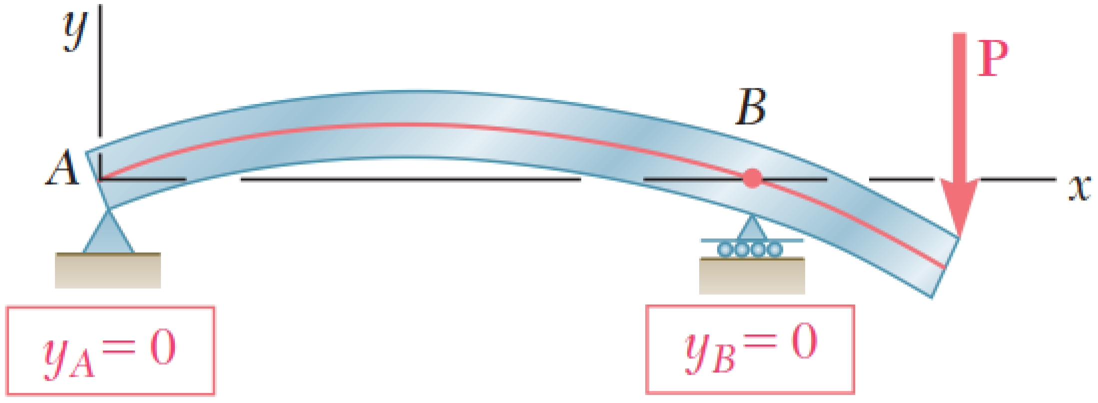
Integrating equation in *x* $\longrightarrow$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1x + C_2$$

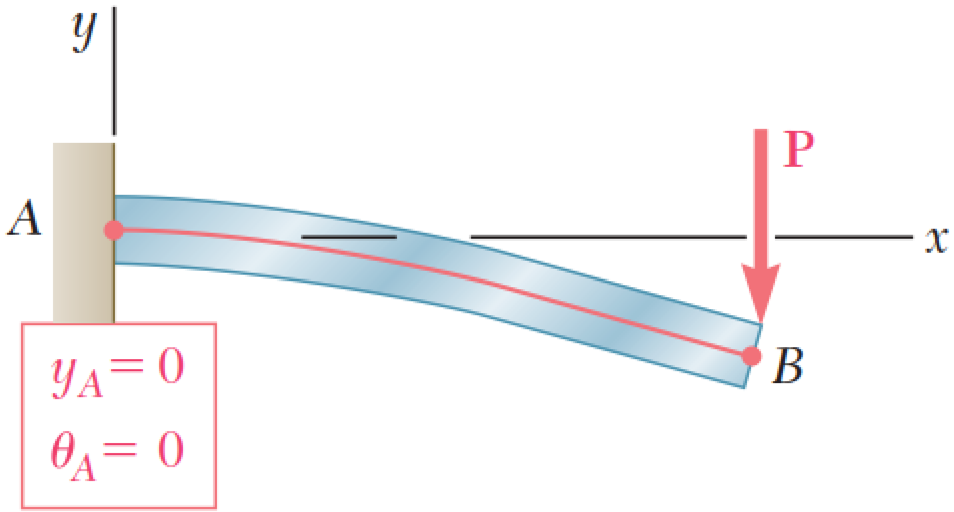
Deflection Equation

The integral constants C_1 and C_2 are determined from the *boundary conditions* or, more precisely, from the conditions imposed on the beam by its supports.

Boundary Conditions:



(b) Overhanging beam

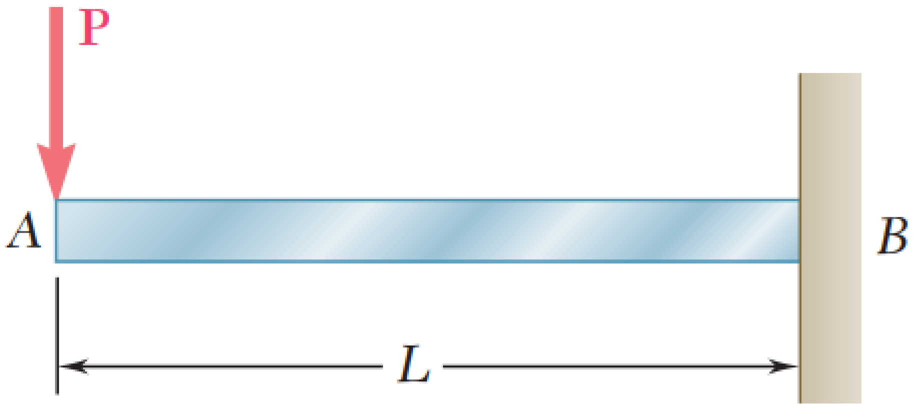


(c) Cantilever beam

$x = x_A$, $y = y_A = 0$
$x = x_B$, $y = y_B = 0$
C_1 and C_2 are obtained from two equations.

$x = x_A$, $y = y_A = 0$
$x = x_A$, x = $\theta_A = 0$
C_1 and C_2 are obtained from two equations.

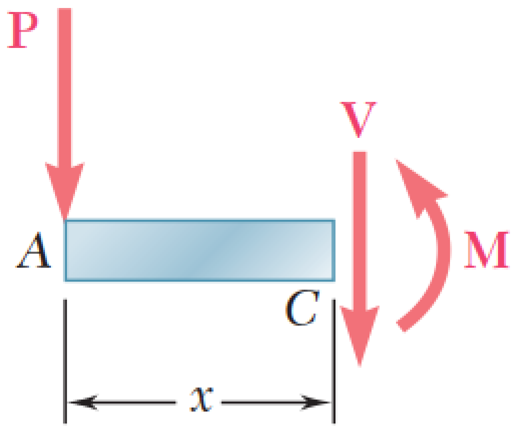
Concept Application 9.1



(a)

Determine the equation of the elastic curve and the deflection and slope at *A*.

Solution



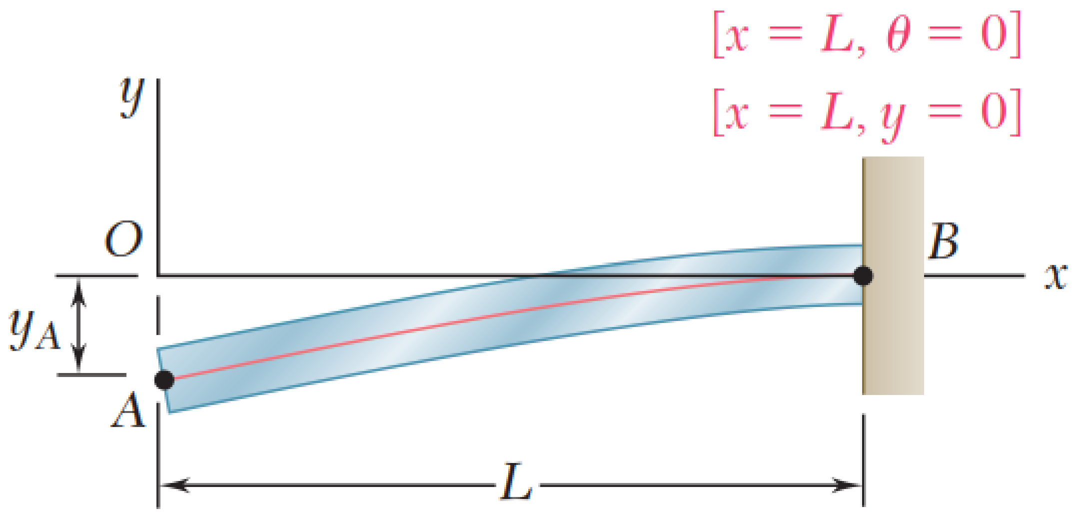
(b)

Using the free-body diagram of the portion AC of the beam, where C is located at a distance x from end A,

$$M = -Px$$

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in *x*, $$EI\frac{dy}{dx} = -\frac{1}{2}Px^2 + C_1$$



(c)

$$x = L \text{ and } \theta = dy/dx = 0$$

$$C_1 = \frac{1}{2}PL^2$$

Solution

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2$$

Integrating in *x*, $EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2$

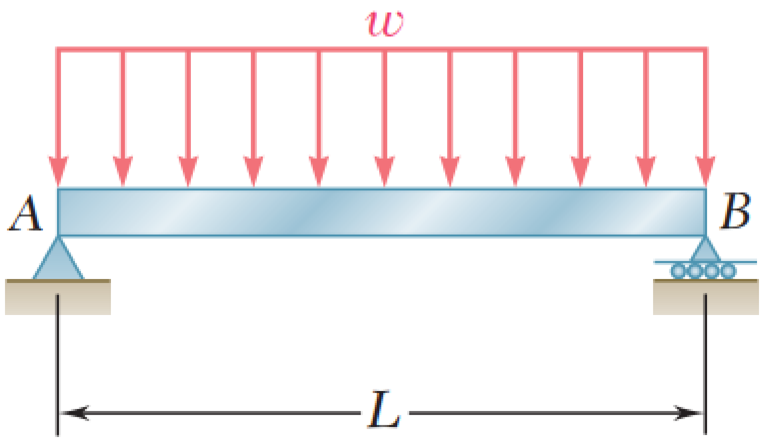
$x = L,\ y = 0 \qquad 0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2 \qquad C_2 = -\tfrac{1}{3}PL^3$

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3 \quad \text{or} \quad y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

The deflection and slope at *A* are obtained by letting $x = 0$

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

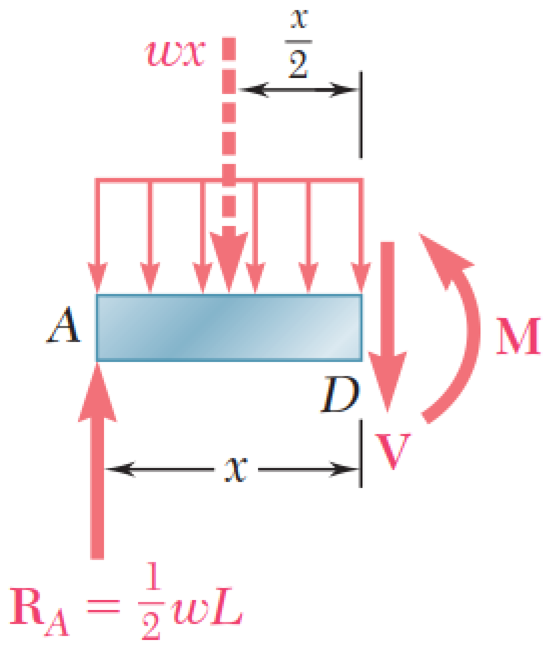
Concept Application 9.2



(a)

Determine the equation of the elastic curve and the maximum deflection of the beam.

Solution



$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$

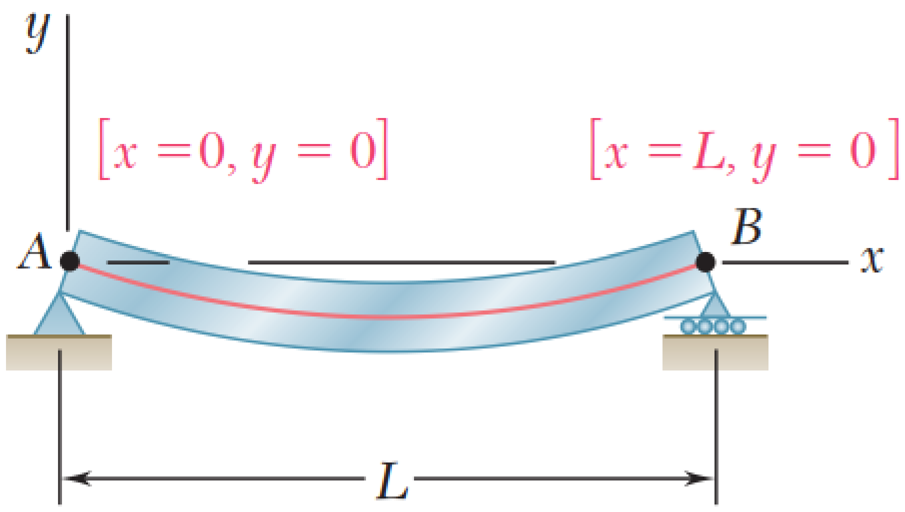
$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

Integrating twice in *x*,

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2$$

Solution

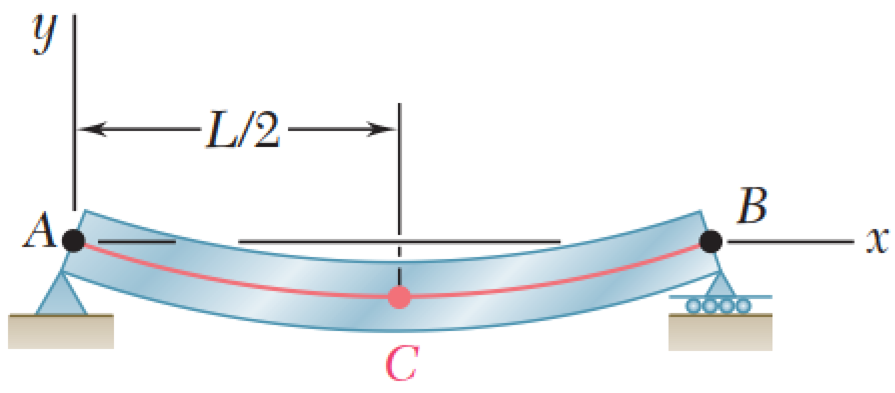


$x = 0$ and $y = 0$ in and obtain $C_2 = 0$

$x = L$ and $y = 0$

$$0 = -\frac{1}{24}wL^4 + \frac{1}{12}wL^4 + C_1L$$

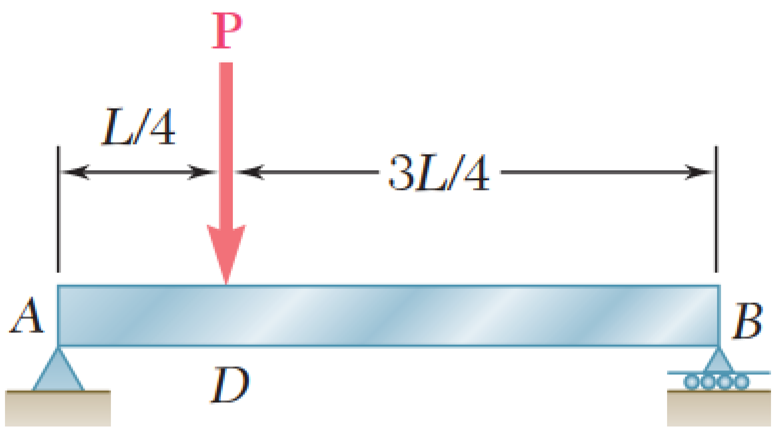
$$C_1 = -\frac{1}{24}wL^3$$



$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

Concept Application 9.3



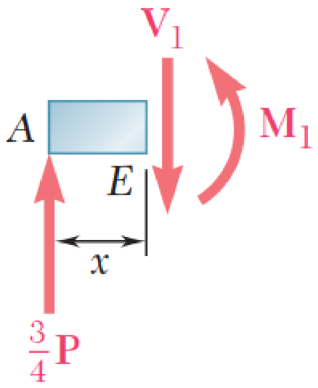
For the prismatic beam and load shown, determine the slope and deflection at point *D*.

Bending moment can be discontinuous at several points in a beam, the *deflection* and the *slope* of the beam *cannot be discontinuous* at any point.

Divide the beam into two portions, *AD* and *DB*, and determine the function *y* (*x*) that defines the elastic curve for each of these portions.

Solution

1. From *A* to *D* ($x < L/4$).



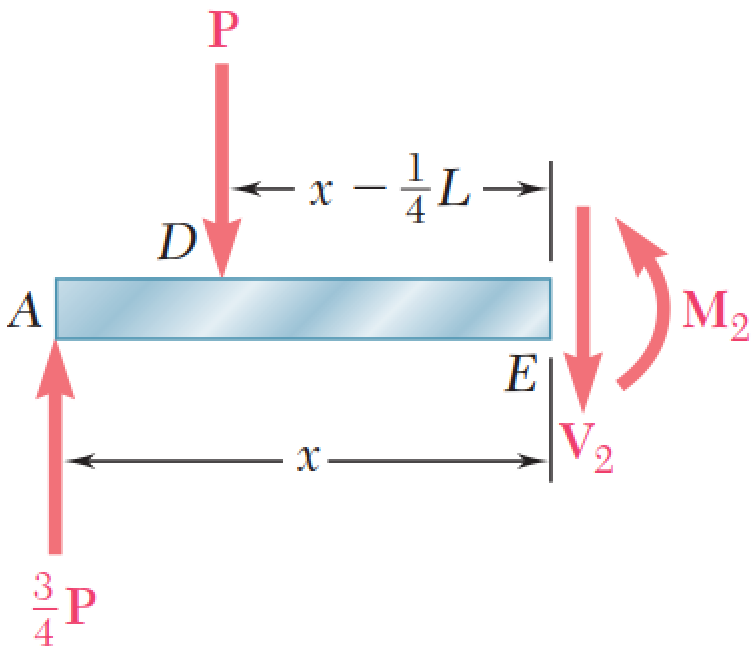
$$M_1 = \frac{3P}{4}x \qquad EI\frac{d^2y_1}{dx^2} = \frac{3}{4}Px$$

$$EI\,\theta_1 = EI\frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1$$

$$EI\,y_1 = \frac{1}{8}Px^3 + C_1x + C_2$$

Solution

2. From *D* to *B* ($x > L/4$).



$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right)$$

$$EI\frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL$$

$$EI\,\theta_2 = EI\frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3$$

$$EI\,y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4$$

Solution

Determination of the Constants of Integration.

$x = 0$ and $y_1 = 0$. $[x = 0, y_1 = 0]$, Eq. (4): $0 = C_2$

$x = L$ and $y_2 = 0$ $[x = L, y_2 = 0]$, Eq. (8): $0 = \frac{1}{12}PL^3 + C_3L + C_4$

at point D requires that $y_1 = y_2$

$\theta_1 = \theta_2$ when $x = L/4$. $[x = L/4, \theta_1 = \theta_2]$, Eqs. (3) and (7):

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3$$

$[x = L/4, y_1 = y_2]$, Eqs. (4) and (8):

$$\frac{PL^3}{512} + C_1\frac{L}{4} = \frac{11PL^3}{1536} + C_3\frac{L}{4} + C_4$$

Solving these equations simultaneously,

$$C_1 = -\frac{7PL^2}{128}, \quad C_2 = 0, \quad C_3 = -\frac{11PL^2}{128}, \quad C_4 = \frac{PL^3}{384}$$

Substituting for C_1 and C_2 into Eqs. (3) and (4), $x \le L/4$ is

$$EI\,\theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128}$$

$$EI\,y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x$$

Letting $x = L/4$ in each of these equations, the slope and deflection at point D are

$$\theta_D = -\frac{PL^2}{32EI} \quad \text{and} \quad y_D = -\frac{3PL^3}{256EI}$$

Determination of the Elastic Curve from the Load Distribution

We know these three relations:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

$$dM/dx = V$$

$$dV/dx = -w$$

Differentiating with respect to x

$$\frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{V(x)}{EI}$$

Differentiating again

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\frac{dV}{dx} = -\frac{w(x)}{EI}$$

when a prismatic beam supports a distributed load $w(x)$, its elastic curve is governed by the fourth-order linear differential equation

$$\frac{d^4y}{dx^4} = -\frac{w(x)}{EI}$$

Determination of the Elastic Curve from the Load Distribution

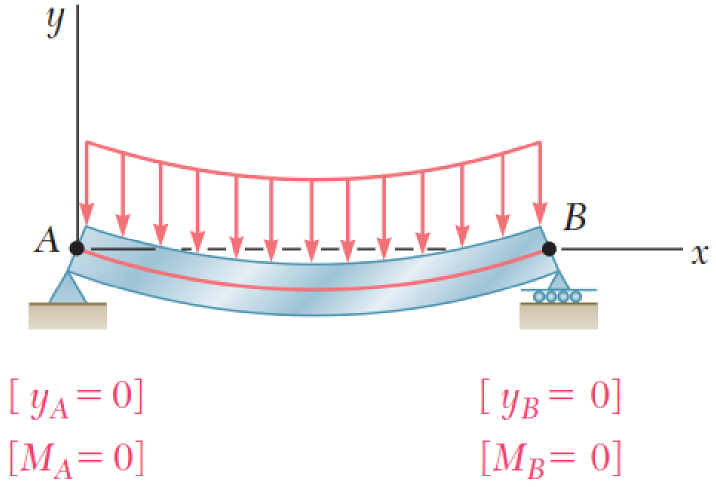
$$EI\frac{d^4y}{dx^4} = -w(x)$$

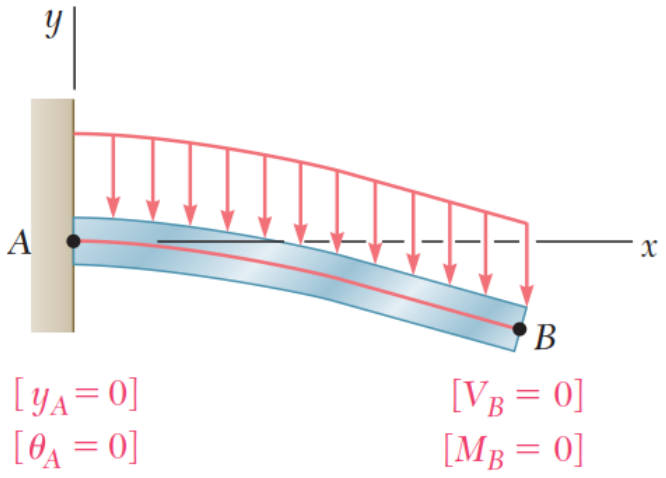
$$EI\frac{d^3y}{dx^3} = V(x) = -\int w(x)\,dx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\int dx\int w(x)\,dx + C_1x + C_2$$

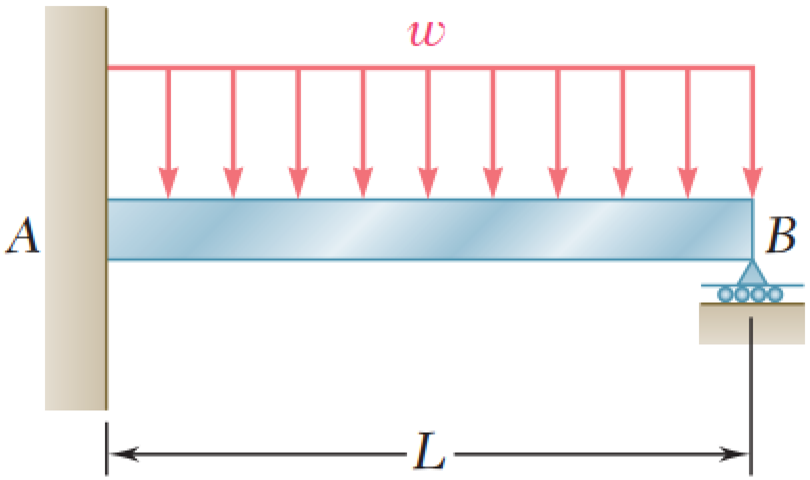
$$EI\frac{dy}{dx} = EI\,\theta(x) = -\int dx\int dx\int w(x)\,dx + \frac{1}{2}C_1x^2 + C_2x + C_3$$

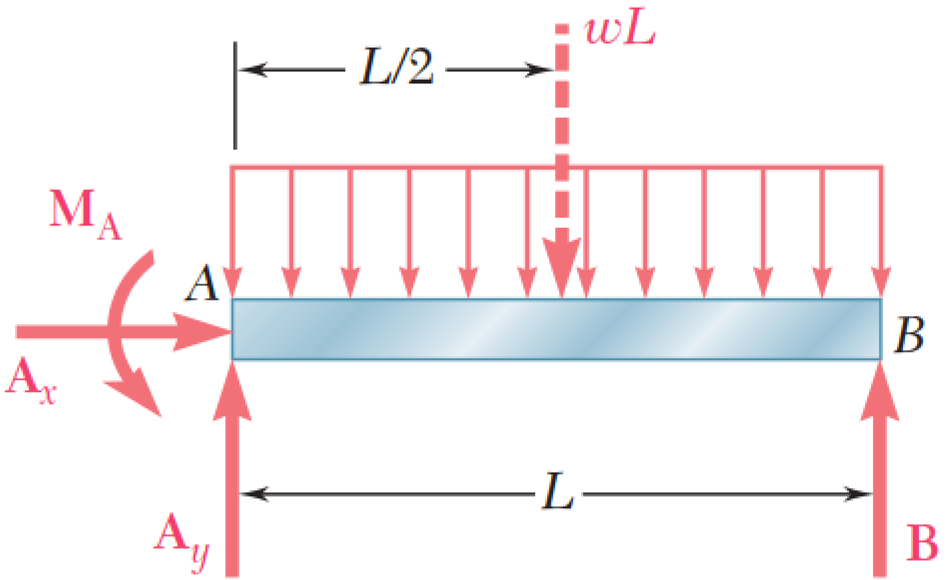
$$EIy(x) = -\int dx\int dx\int dx\int w(x)\,dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$





STATICALLY INDETERMINATE BEAMS

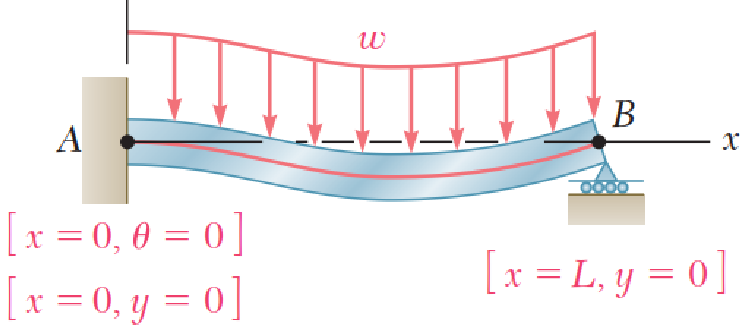




$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M_A = 0$$

Since only A_x can be determined from these equations, the beam is *statically indeterminate*.

The reactions can be obtained by considering the *deformations* of the structure.



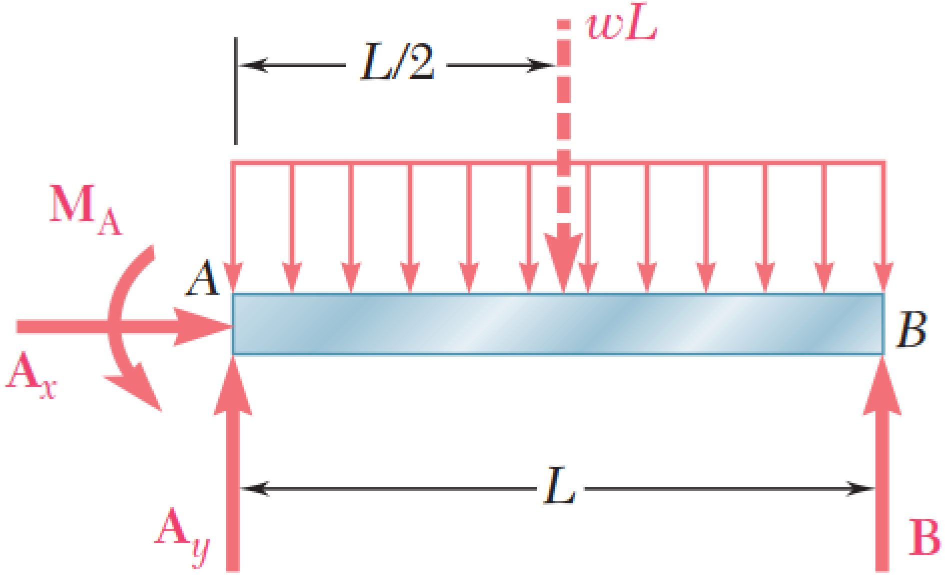
Concept Application 9.5

Equilibrium Equations. From the free-body diagram

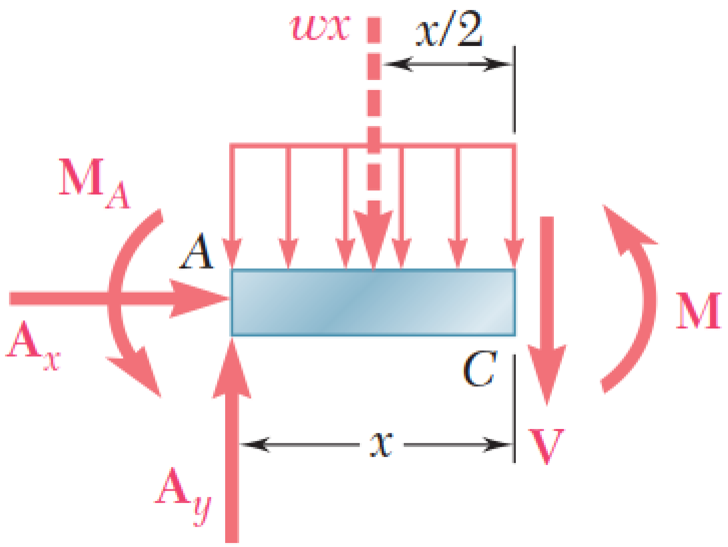
$$\overset{+}{\rightarrow} \Sigma F_x = 0: \qquad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \qquad A_y + B - wL = 0$$

$$+\circlearrowleft \Sigma M_A = 0: \qquad M_A + BL - \tfrac{1}{2} wL^2 = 0$$



Concept Application 9.5



Equation of Elastic Curve. Draw the free-body diagram of a portion of beam *AC* to obtain

$$+\circlearrowleft \Sigma M_C = 0: \qquad M + \tfrac{1}{2}wx^2 + M_A - A_y x = 0$$

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + A_y x - M_A$$

$$EI\,\theta = EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{2}A_y x^2 - M_A x + C_1$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{6}A_y x^3 - \frac{1}{2}M_A x^2 + C_1 x + C_2$$

Concept Application 9.5

Boundary Conditions

$$x = 0,\ \theta = 0$$

$$x = 0,\ y = 0$$

$$\Rightarrow \quad C_1 = C_2 = 0.$$

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{6}A_y x^3 - \tfrac{1}{2}M_A x^2$$

$$x = L \quad y = 0 \quad \Rightarrow \quad 0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{6}A_y L^3 - \tfrac{1}{2}M_A L^2$$

$$3M_A - A_y L + \tfrac{1}{4}wL^2 = 0$$

Solving this equation simultaneously with the three equilibrium equations, the reactions at the supports are

$$A_x = 0 \qquad A_y = \tfrac{5}{8}wL \qquad M_A = \tfrac{1}{8}wL^2 \qquad B = \tfrac{3}{8}wL$$

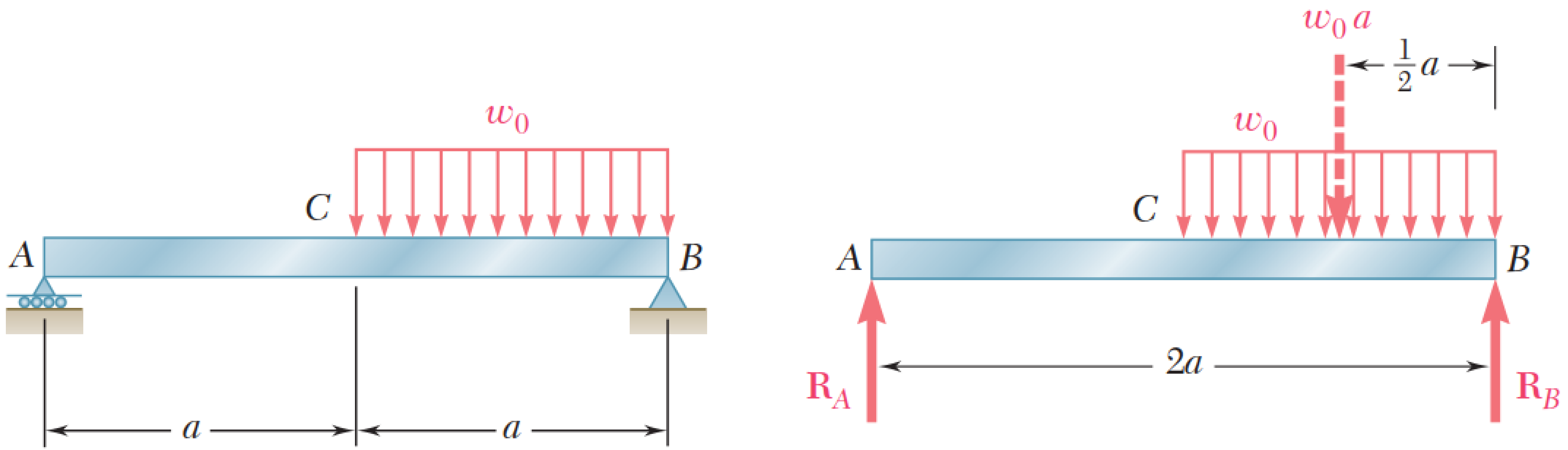
SINGULARITY FUNCTIONS TO DETERMINE SLOPE AND DEFLECTION

The integration method provides a convenient and effective way of determining the slope and deflection at any point of a prismatic beam, *as long as the bending moment can be represented by a single analytical function* $M(x)$.

However, when the loading of the beam needs more than one functions to represent the bending moment over the entire length.

Singularity functions makes it possible to represent the shear V and bending moment M with single mathematical expressions.

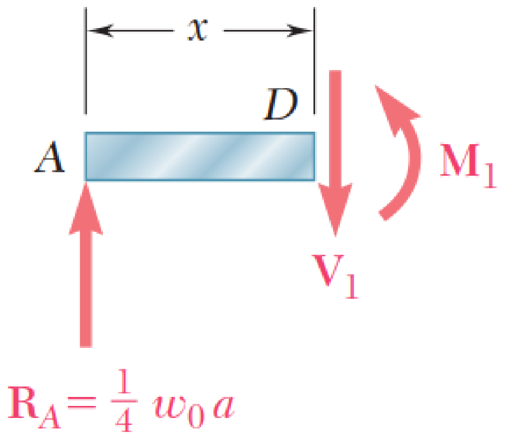
SINGULARITY FUNCTIONS TO DETERMINE SLOPE AND DEFLECTION Cont.



$$+\circlearrowleft \Sigma M_B = 0: \qquad (w_0 a)(\tfrac{1}{2}a) - R_A(2a) = 0 \qquad R_A = \tfrac{1}{4} w_0 a$$

SINGULARITY FUNCTIONS TO DETERMINE SLOPE AND DEFLECTION Cont.

Cut the beam at a point *D* between *A* and *C*.

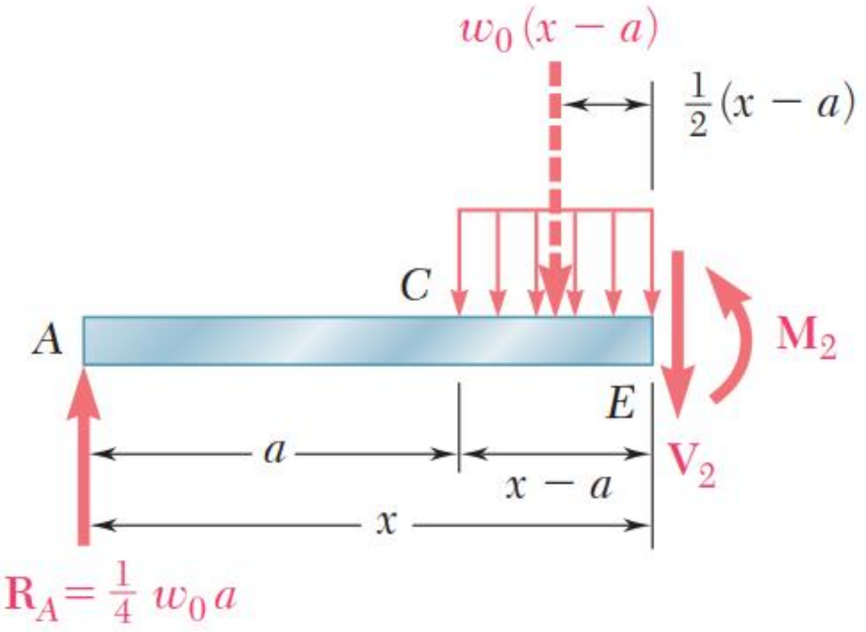


$$0 < x < a$$

$$V_1(x) = \tfrac{1}{4}w_0a$$

$$M_1(x) = \tfrac{1}{4}w_0ax$$

Cutting the beam at a point *E* between *C* and *B*



$$a < x < 2a$$

$$V_2(x) = \tfrac{1}{4}w_0a - w_0(x - a)$$

$$M_2(x) = \tfrac{1}{4}w_0ax - \tfrac{1}{2}w_0(x - a)^2$$

SINGULARITY FUNCTIONS TO DETERMINE SLOPE AND DEFLECTION Cont.

$V_1(x)$ and $V_2(x)$ can be represented by the single function

$$V(x) = \frac{1}{4}w_0 a - w_0\langle x - a\rangle$$

the brackets < > should be replaced by ordinary parentheses ( ) when $x \geq a$ and by zero when $x < a$. Similarly

$$M(x) = \frac{1}{4}w_0 ax - \frac{1}{2}w_0\langle x - a\rangle^2$$

The function within the brackets < > can be differentiated or integrated as if the brackets were replaced with ordinary parentheses.

SINGULARITY FUNCTIONS TO DETERMINE SLOPE AND DEFLECTION Cont.

Instead of calculating the bending moment from free-body diagrams

$$M(x) - M(0) = \int_0^x V(x)\,dx = \int_0^x \tfrac{1}{4} w_0 a\,dx - \int_0^x w_0 \langle x - a \rangle dx$$

After integration and observing that $M(0) = 0$,

$$M(x) = \tfrac{1}{4} w_0 a x - \tfrac{1}{2} w_0 \langle x - a \rangle^2$$

Furthermore, using the same convention, the distributed load at any point of the beam can be expressed as

$$w(x) = w_0 \langle x - a \rangle^0$$

SINGULARITY FUNCTIONS TO DETERMINE SLOPE AND DEFLECTION Cont.

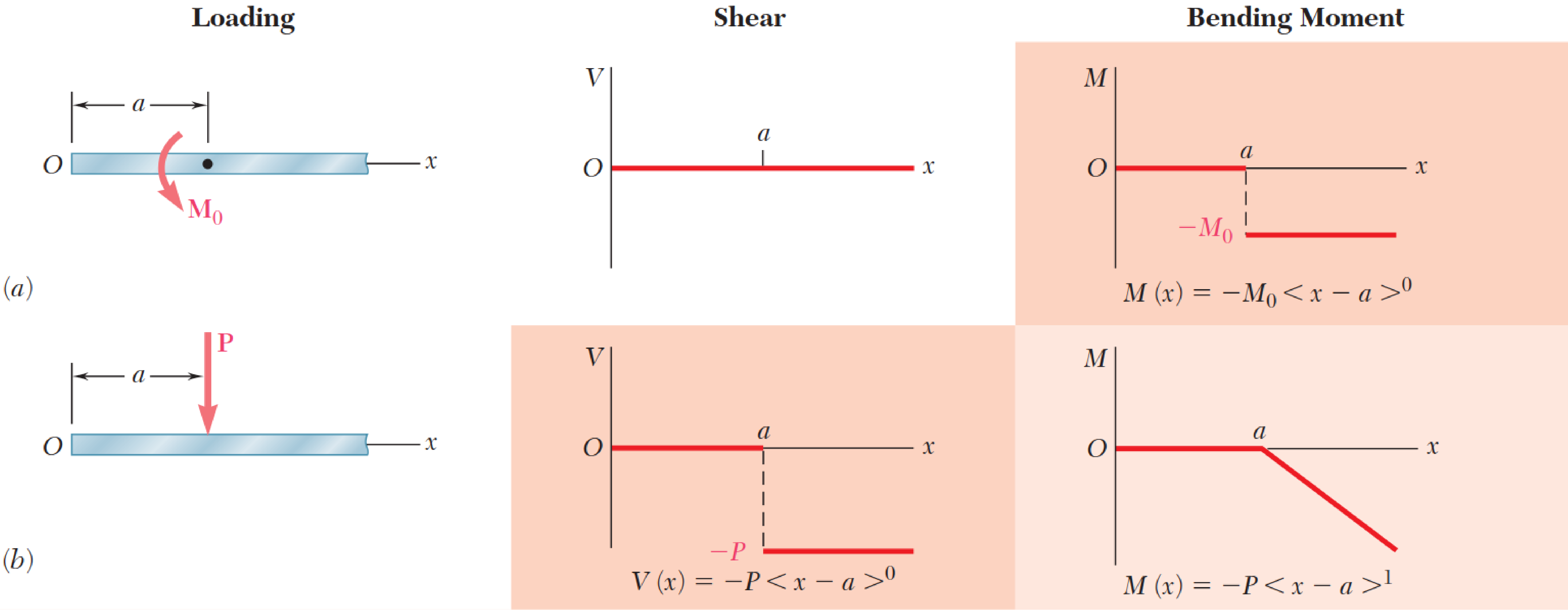
$$V(x) - V(0) = -\int_0^x w(x)\,dx = -\int_0^x w_0\langle x - a\rangle^0\,dx$$

$$V(x) - \tfrac{1}{4}w_0 a = -w_0\langle x - a\rangle^1$$

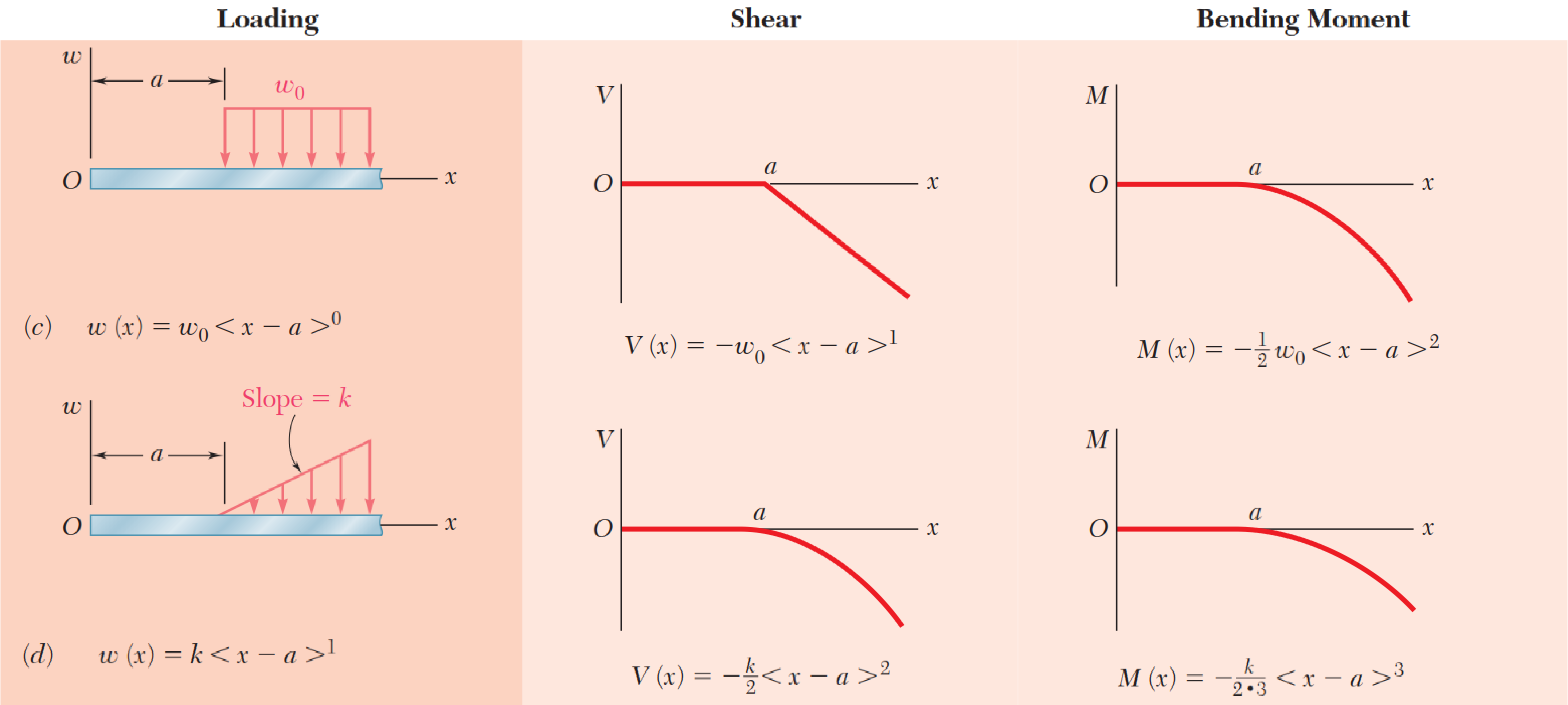
The expressions $\langle x - a\rangle^0, \langle x - a\rangle, \langle x - a\rangle^2$ are called *singularity functions*. For $n \geq 0$,

$$\langle x - a\rangle^n = \begin{cases} (x - a)^n & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases}$$

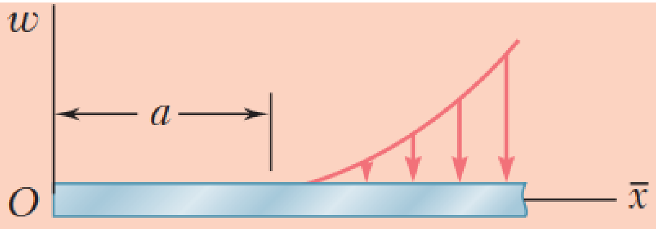
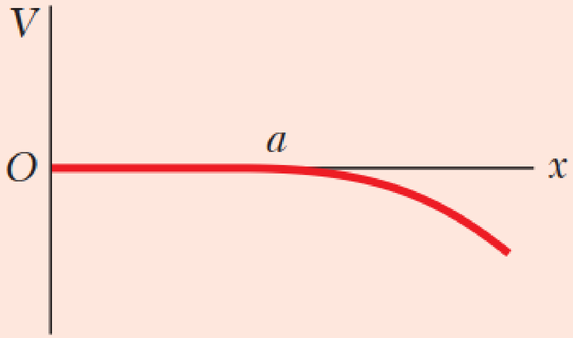
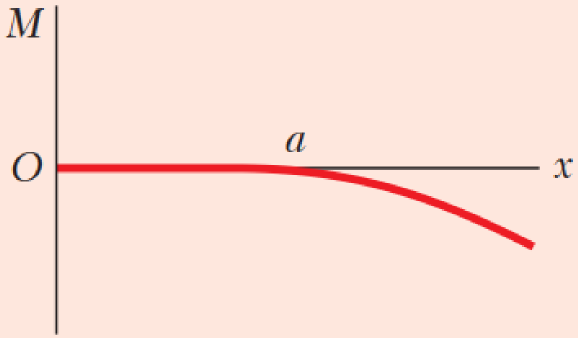
Basic loadings and corresponding shears and bending moments expressed in terms of singularity functions

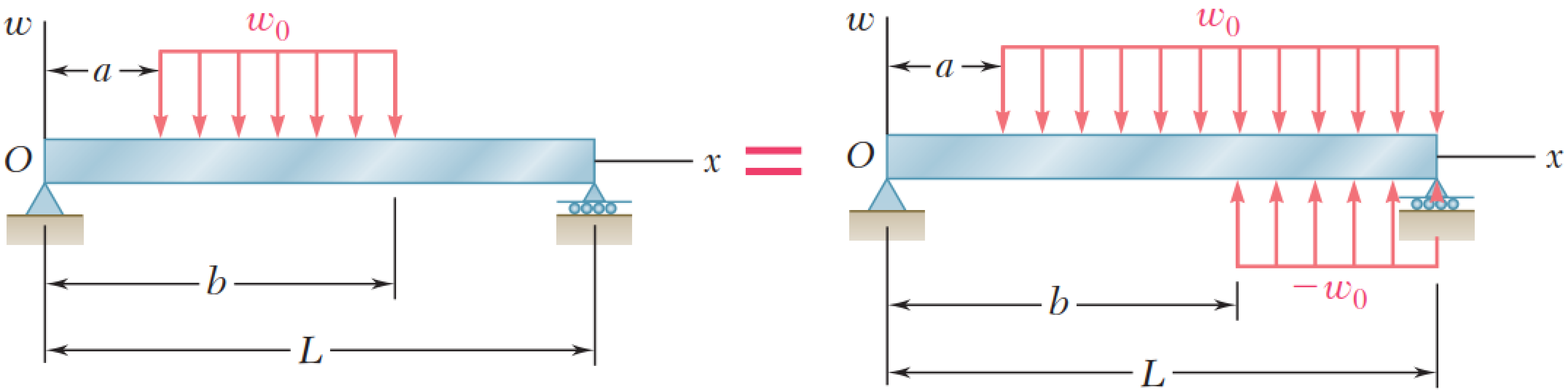


Basic loadings and corresponding shears and bending moments expressed in terms of singularity functions



Basic loadings and corresponding shears and bending moments expressed in terms of singularity functions

Loading	Shear	Bending Moment
		
(e) $w(x) = k < x - a >^{n}$	$V(x) = -\frac{k}{n+1} < x - a >^{n+1}$	$M(x) = -\frac{k}{(n+1)(n+2)} < x - a >^{n+2}$

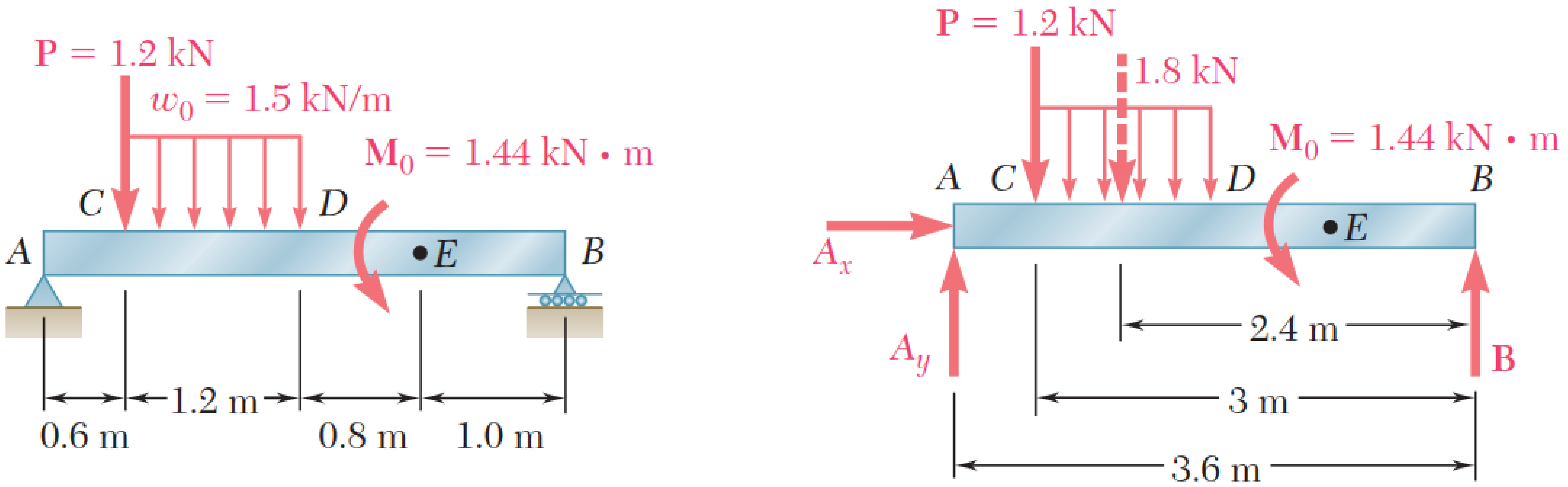


$$w(x) = w_0 < x - a >^0 - w_0 < x - b >^0$$

Fig. 5.17 Use of open-ended loadings to create a closed-ended loading.

Concept Application 9.6

$E = 200 \text{ GPa and } I = 6.87 \times 10^{-6} \text{ m}^4$

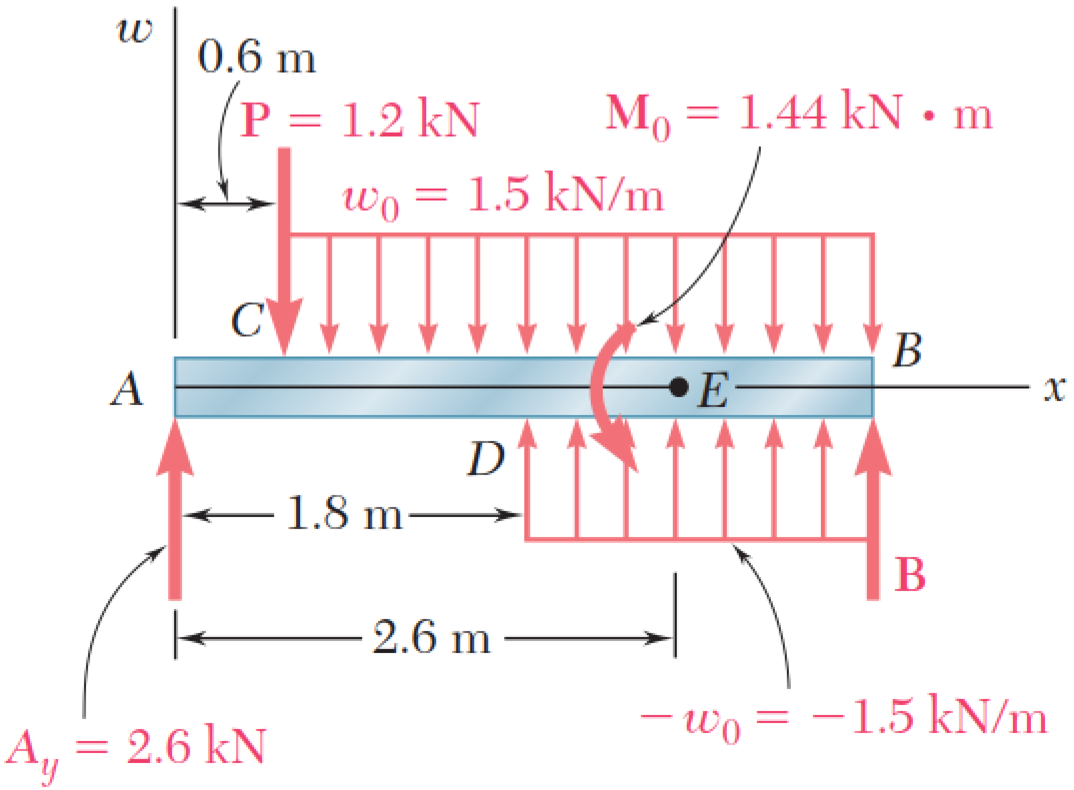


$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -A_y(3.6 \text{ m}) + (1.2 \text{ kN})(3 \text{ m})$$

$$+(1.8 \text{ kN})(2.4 \text{ m}) + 1.44 \text{ kN·m} = 0$$

$$A_y = 2.60 \text{ kN}$$



$$w(x) = +w_0\langle x - 0.6\rangle^0 - w_0\langle x - 1.8\rangle^0$$

$$V(x) = -w_0\langle x - 0.6\rangle^1 + w_0\langle x - 1.8\rangle^1 + A_y - P\langle x - 0.6\rangle^0$$

$$M(x) = -\tfrac{1}{2}w_0\langle x - 0.6\rangle^2 + \tfrac{1}{2}w_0\langle x - 1.8\rangle^2 + A_y x - P\langle x - 0.6\rangle^1 - M_0\langle x - 2.6\rangle^0$$

$$V(x) = -1.5\langle x - 0.6\rangle^1 + 1.5\langle x - 1.8\rangle^1 + 2.6 - 1.2\langle x - 0.6\rangle^0$$

$$M(x) = -0.75\langle x - 0.6\rangle^2 + 0.75\langle x - 1.8\rangle^2 + 2.6x - 1.2\langle x - 0.6\rangle^1 - 1.44\langle x - 2.6\rangle^0$$

$$EI\theta = -0.25\langle x - 0.6\rangle^3 + 0.25\langle x - 1.8\rangle^3 + 1.3x^2 - 0.6\langle x - 0.6\rangle^2 - 1.44\langle x - 2.6\rangle^1 + C_1$$

$$EIy = -0.0625\langle x - 0.6\rangle^4 + 0.0625\langle x - 1.8\rangle^4 + 0.4333x^3 - 0.2\langle x - 0.6\rangle^3 - 0.72\langle x - 2.6\rangle^2 + C_1x + C_2$$

$x = 0,\ y = 0 \qquad C_2 = 0$

$x = 3.6,\ y = 0 \qquad C_1 = -2.692$

$x = x_D = 1.8$ m, we find that the deflection at point *D*

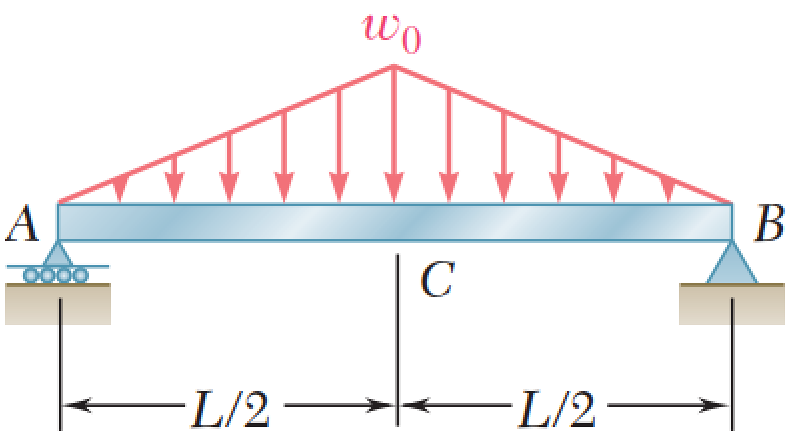
$$EIy_D = -0.0625\langle 1.2\rangle^4 + 0.0625\langle 0\rangle^4$$
$$+ 0.4333(1.8)^3 - 0.2\langle 1.2\rangle^3 - 0.72\langle -0.8\rangle^2 - 2.692(1.8)$$

$$EIy_D = -0.0625(1.2)^4 + 0.0625(0)^4$$
$$+ 0.4333(1.8)^3 - 0.2(1.2)^3 - 0 - 2.692(1.8) = -2.794$$

Recalling the given numerical values of *E* and *I*,

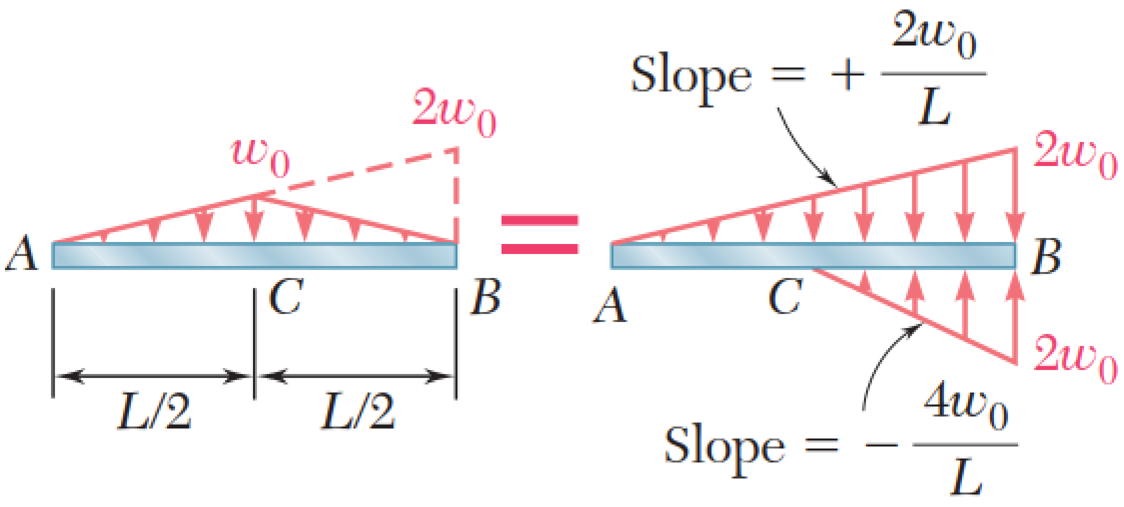
$$(200\text{ GPa})(6.87 \times 10^{-6}\text{ m}^4)y_D = -2.794\text{ kN}\cdot\text{m}^3$$
$$y_D = -13.64 \times 10^{-3}\text{ m} = -2.03\text{ mm}$$

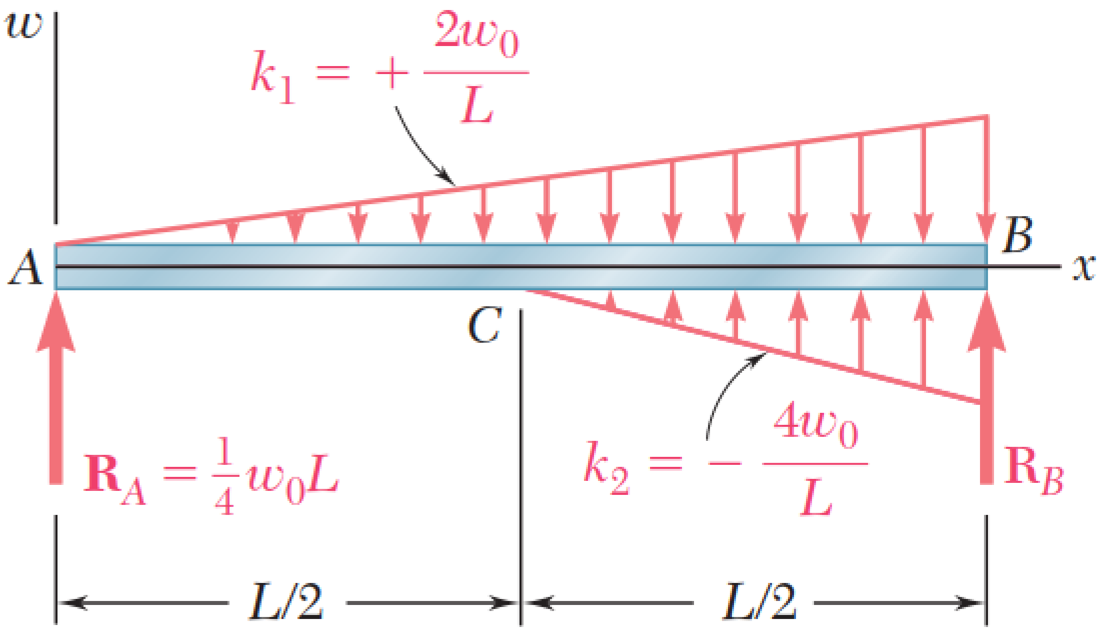
Sample Problem 9.4



For the prismatic beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at *A*, (*c*) the maximum deflection.

Solution





Reactions. Due to symmetry, each reaction is

$$\tfrac{1}{4}w_0L$$

Solution

$$w(x) = k_1 x + k_2 \langle x - \tfrac{1}{2}L \rangle = \frac{2w_0}{L} x - \frac{4w_0}{L} \langle x - \tfrac{1}{2}L \rangle$$

$$V(x) = -\frac{w_0}{L} x^2 + \frac{2w_0}{L} \langle x - \tfrac{1}{2}L \rangle^2 + \tfrac{1}{4} w_0 L$$

$$M(x) = -\frac{w_0}{3L} x^3 + \frac{2w_0}{3L} \langle x - \tfrac{1}{2}L \rangle^3 + \tfrac{1}{4}\, w_0 L x$$

Equation of the Elastic Curve

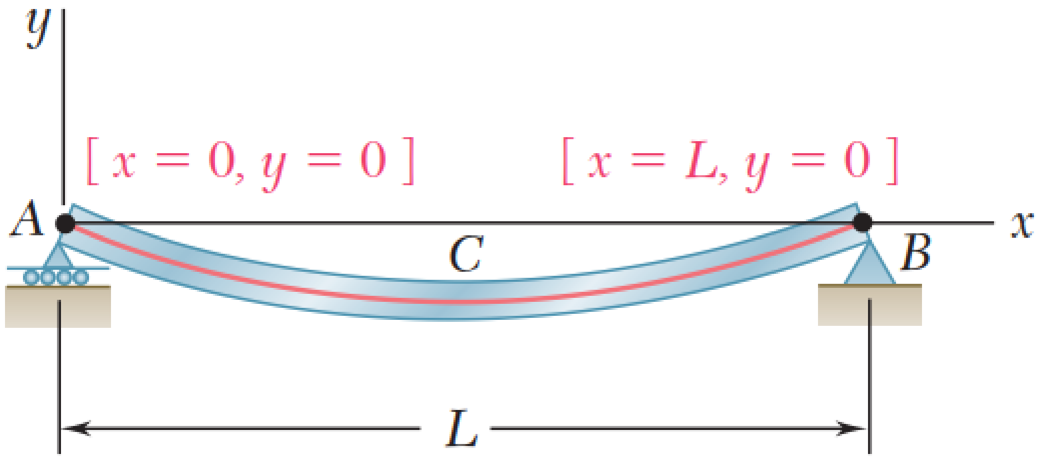
$$EI\frac{d^2y}{dx^2} = -\frac{w_0}{3L}x^3 + \frac{2w_0}{3L}\langle x - \tfrac{1}{2}L\rangle^3 + \tfrac{1}{4}w_0Lx$$

integrating twice in x,

$$EI\,\theta = -\frac{w_0}{12L}x^4 + \frac{w_0}{6L}\langle x - \tfrac{1}{2}L\rangle^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EI\,y = -\frac{w_0}{60L}x^5 + \frac{w_0}{30L}\langle x - \tfrac{1}{2}L\rangle^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

a. Boundary Conditions



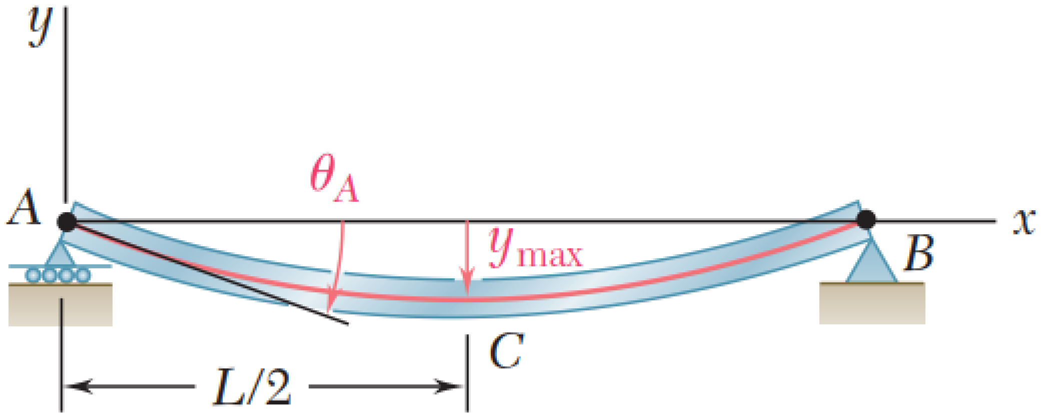
$[x = 0, y = 0] \qquad C_2 = 0$

$[x = L, y = 0] \qquad 0 = -\frac{w_0 L^4}{60} + \frac{w_0}{30L}\left(\frac{L}{2}\right)^5 + \frac{w_0 L^4}{24} + C_1 L \qquad C_1 = -\frac{5}{192} w_0 L^3$

$$EI\,\theta = -\frac{w_0}{12L}x^4 + \frac{w_0}{6L}\langle x - \tfrac{1}{2}L\rangle^4 + \frac{w_0 L}{8}x^2 - \frac{5}{192}w_0 L^3$$

$$EI\,y = -\frac{w_0}{60L}x^5 + \frac{w_0}{30L}\langle x - \tfrac{1}{2}L\rangle^5 + \frac{w_0 L}{24}x^3 - \frac{5}{192}w_0 L^3 x$$

b. Slope at A



$$x = 0$$

$$EI\,\theta_A = -\frac{5}{192}w_0L^3$$

$$\theta_A = \frac{5w_0L^3}{192EI} \curvearrowleft$$

c. Maximum Deflection

The maximum deflection occurs at point *C* $x = \frac{1}{2}L$

$$EI\,y_{max} = w_0L^4\left[-\frac{1}{60(32)} + 0 + \frac{1}{24(8)} - \frac{5}{192(2)}\right] = -\frac{w_0L^4}{120}$$

$$y_{max} = \frac{w_0L^4}{120EI}\downarrow$$

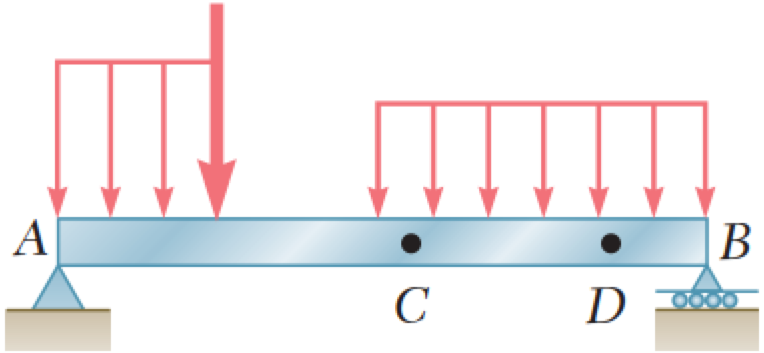
MOMENT-AREA THEOREMS

General Principles

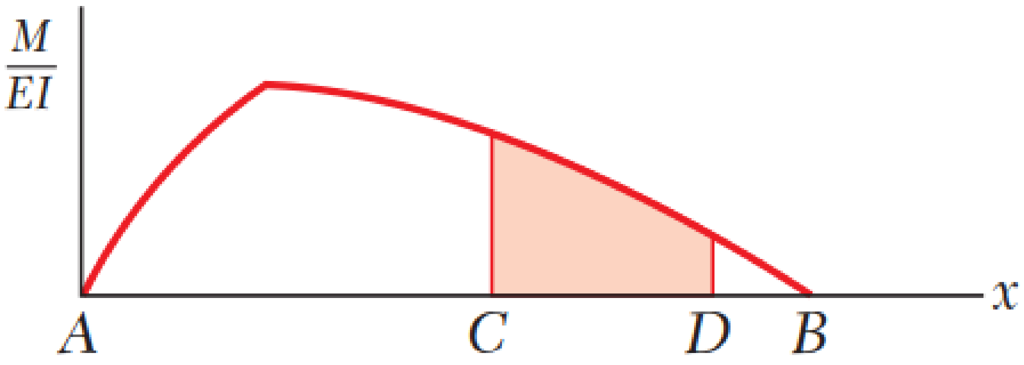
We used a mathematical method based on the integration of a differential equation to determine the deflection and slope of a beam at any given point.

In this section you will see how geometric properties of the elastic curve can be used to determine the deflection and slope of a beam at a specific point.

First Moment-Area Theorem.



Draw the diagram representing the variation along the beam of *M*/*EI* obtained by dividing the bending moment *M* by the flexural rigidity *EI*



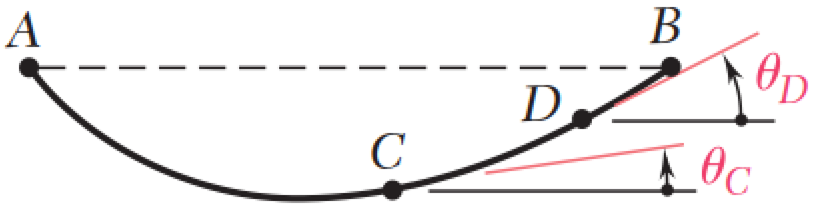
Recalling that $dy/dx = \theta$,

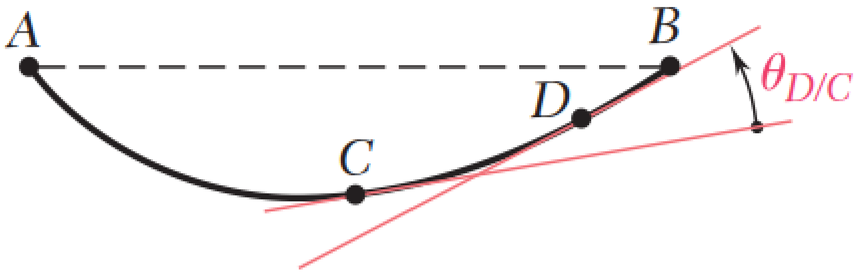
$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI} \qquad \text{or} \qquad d\theta = \frac{M}{EI}\,dx$$

Consider two arbitrary points *C* and *D* on the beam and integrate both members from *C* to *D* :

$$\int_{\theta_C}^{\theta_D} d\theta = \int_{x_C}^{x_D} \frac{M}{EI}\,dx \qquad \text{or} \qquad \theta_D - \theta_C = \int_{x_C}^{x_D} \frac{M}{EI}\,dx$$

Where θ_C and θ_D indicate the slope at *C* and *D*. But the righthand member of equation represents the area under the *M*/*EI* diagram between *C* and *D*, while the left-hand member is the angle between the tangents to the elastic curve at *C* and *D*, $\theta_{D/C}$



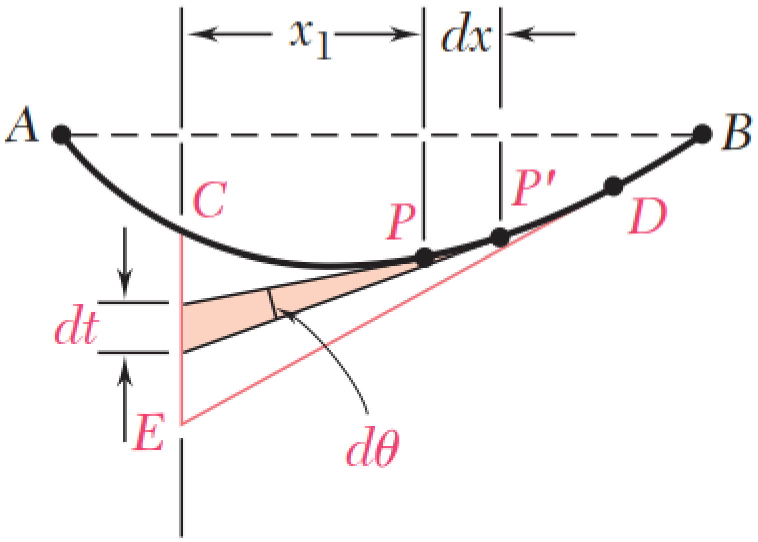


$$\theta_{D/C} = \text{area under } M/EI \text{ diagram between } C \text{ and } D$$

This is the *first moment-area theorem*.

Note that $\theta_{D/C}$ and the area under the *M/EI* diagram have the same sign. This positive area (i.e., located above the *x* axis) corresponds to a counterclockwise rotation of the tangent to the elastic curve moving from *C* to *D*, and a negative area corresponds to a clockwise rotation.

Second Moment-Area Theorem.



Now consider two points *P* and *P'* located between *C* and *D* at a distance *dx* from each other.

The tangents to the elastic curve drawn at *P* and *P'* intercept a segment with a length *dt* on the vertical through point *C*.

Since the slope θ at *P* and the angle $d\theta$ formed by the tangents at *P* and *P'* are both small quantities, *dt* is assumed to be equal to the arc of the circle of radius x_1 subtending the angle $d\theta$. Therefore,

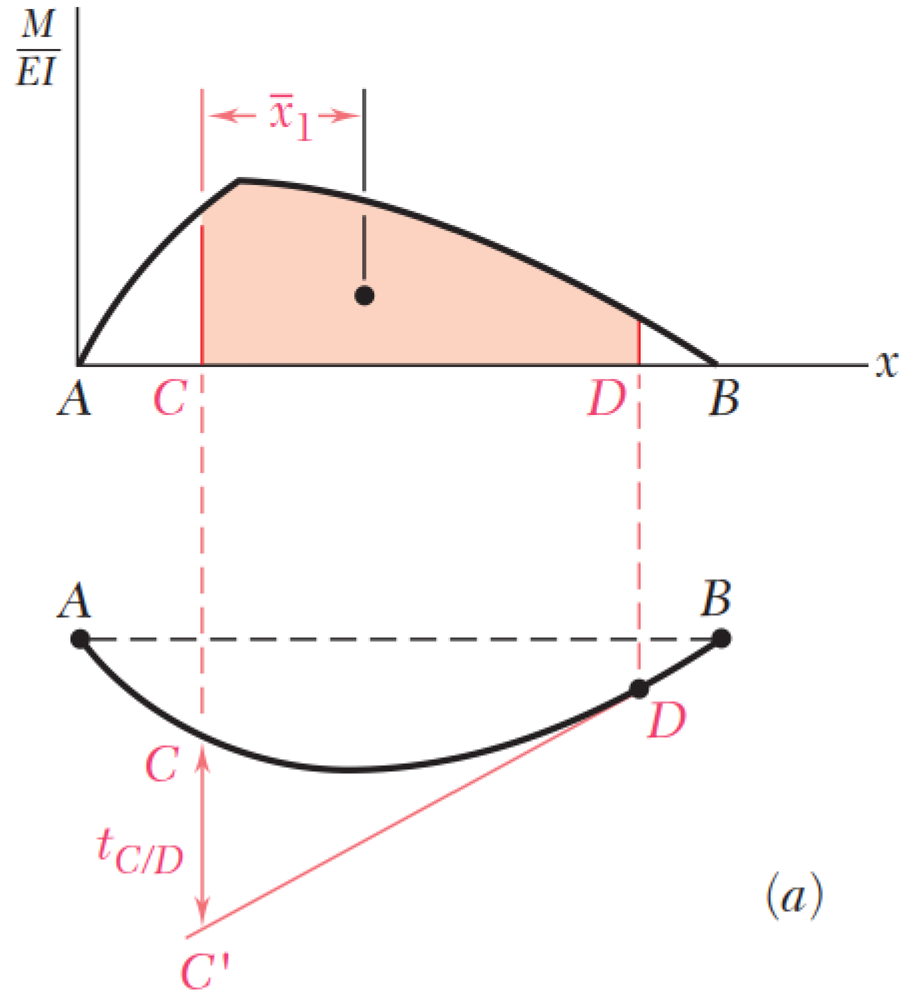
$$dt = x_1\, d\theta$$

$$dt = x_1 \frac{M}{EI}\, dx$$

$$t_{C/D} = \int_{x_C}^{x_D} x_1 \frac{M}{EI}\, dx$$

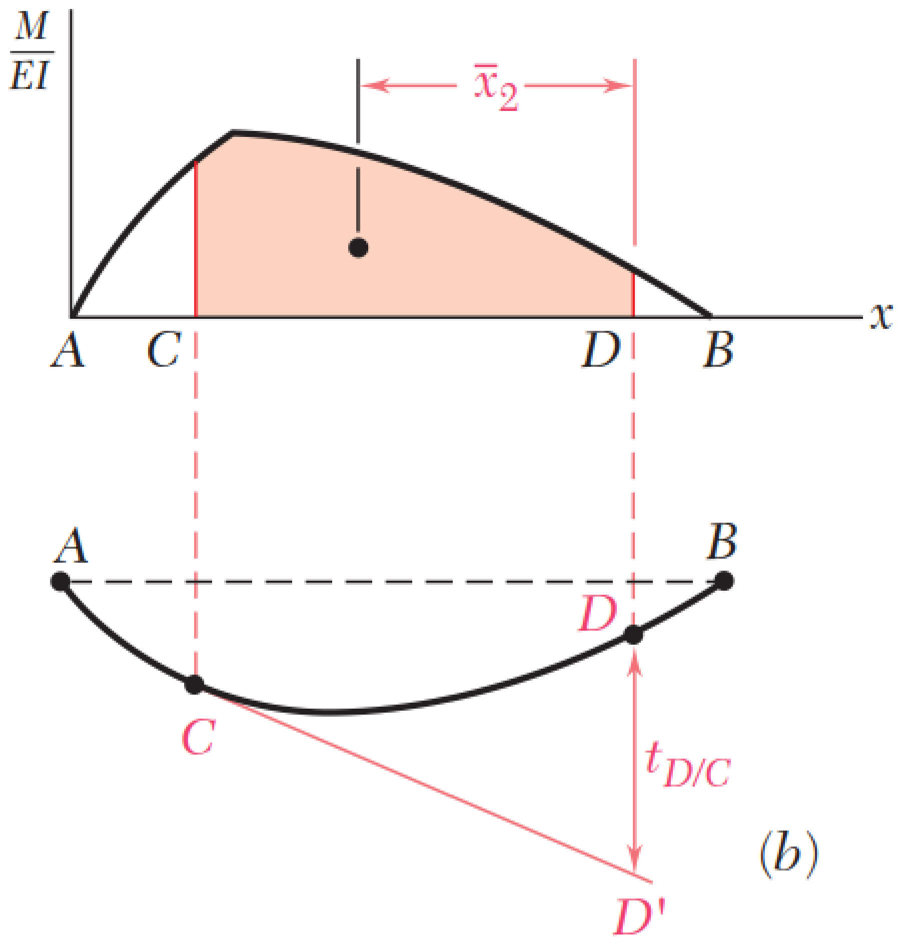
$$t_{C/D} = (\text{area between } C \text{ and } D)\, \bar{x}_1$$

Second Moment-Area Theorem.



$$t_{C/D} = (\text{area between } C \text{ and } D)\, \bar{x}_1$$

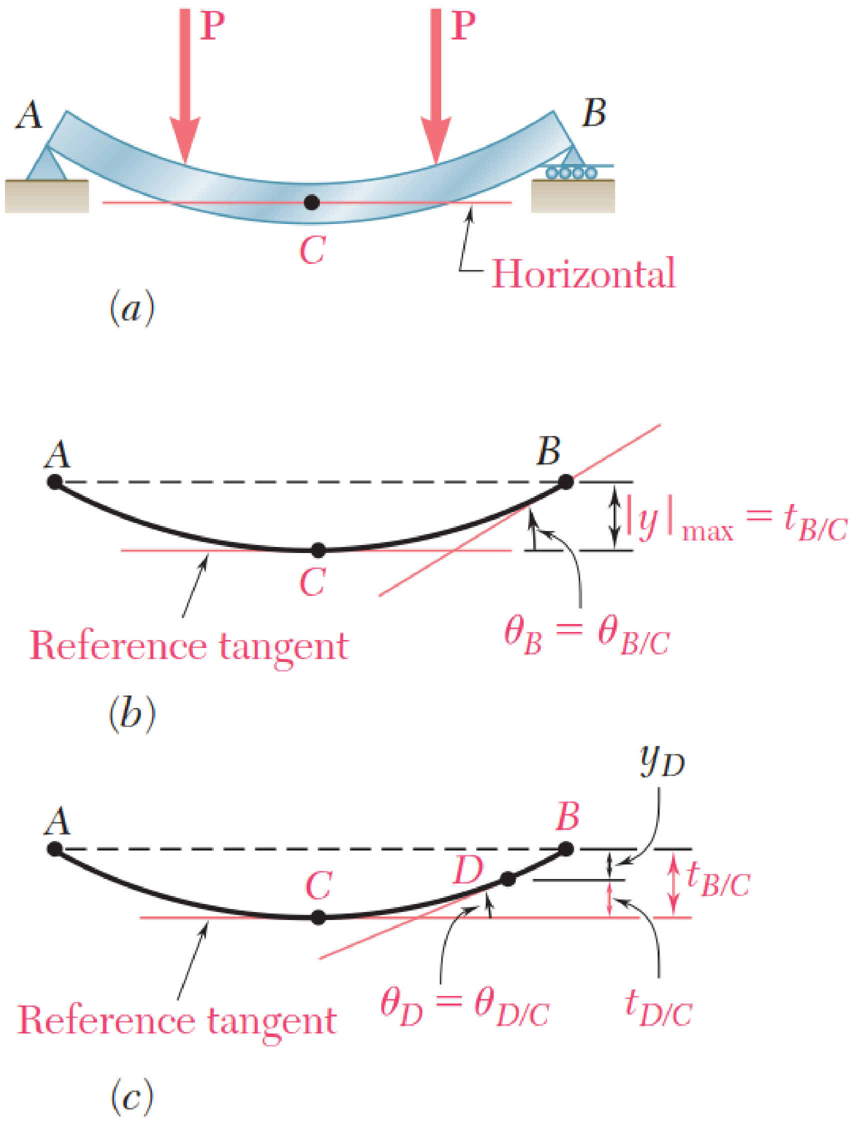
Second Moment-Area Theorem.

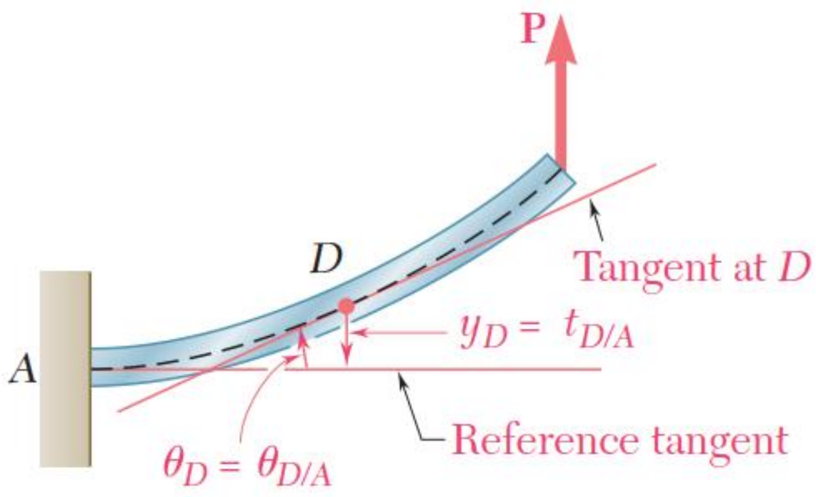


(b)

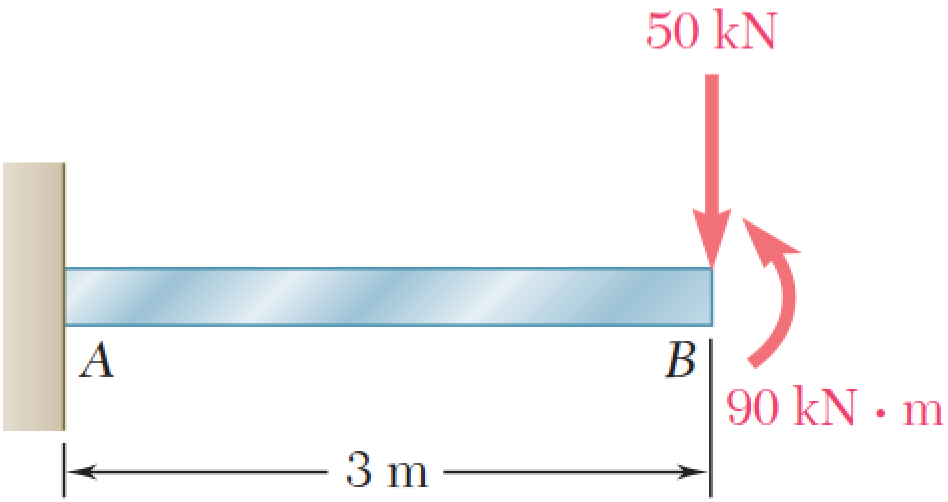
$$t_{D/C} = (\text{area between } C \text{ and } D)\,\bar{x}_2$$

Cantilever Beams and Beams with Symmetric Loadings

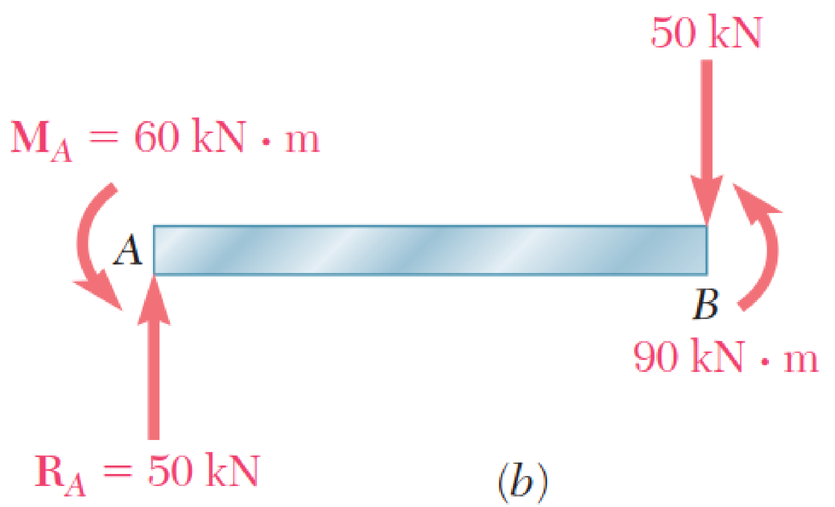




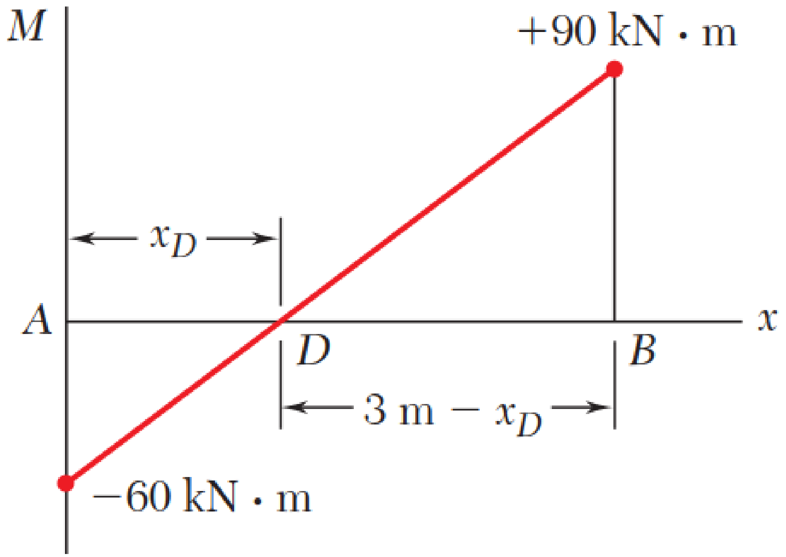
Concept Application 9.9



Determine the slope and deflection at end *B* of the prismatic cantilever beam *AB* when it is loaded as shown, knowing that the flexural rigidity of the beam is EI = 10 MN.m^2



(b)



$$\frac{x_D}{60} = \frac{3 - x_D}{90} = \frac{3}{150} \qquad x_D = 1.2\text{ m}$$

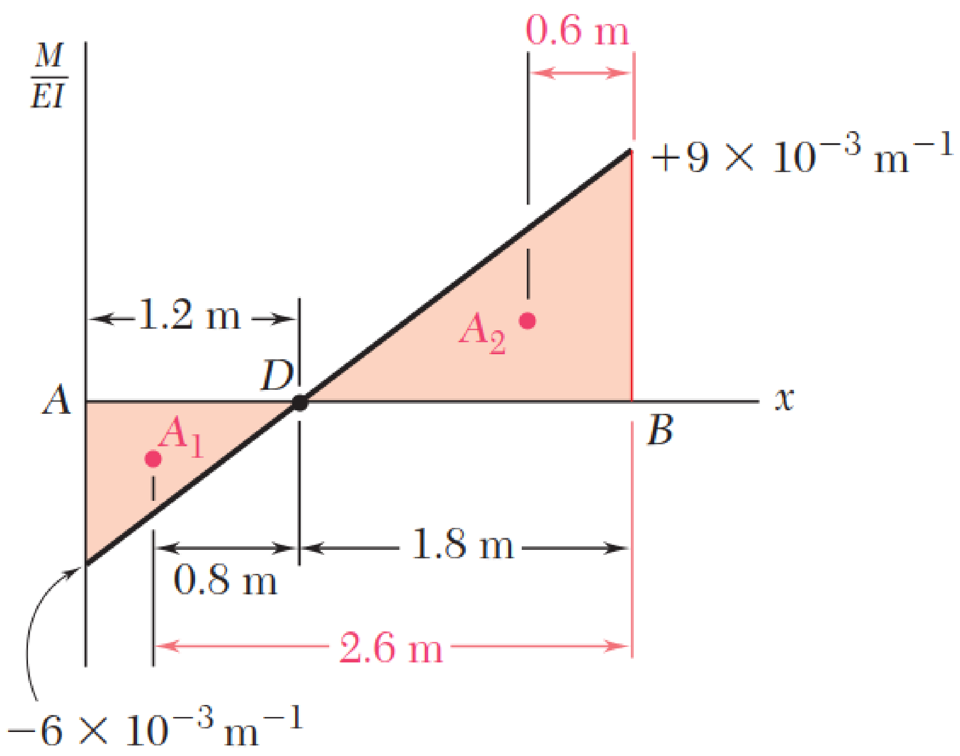
Use the first moment-area theorem to obtain

$$\begin{aligned}\theta_{B/A} = \theta_B - \theta_A &= \text{area from } A \text{ to } B = A_1 + A_2 \\ &= -\tfrac{1}{2}(1.2\text{ m})(6 \times 10^{-3}\text{ m}^{-1}) + \tfrac{1}{2}(1.8\text{ m})(9 \times 10^{-3}\text{ m}^{-1}) \\ &= -3.6 \times 10^{-3} + 8.1 \times 10^{-3} \\ &= +4.5 \times 10^{-3}\text{ rad}\end{aligned}$$

and, since $\theta_A = 0$,

$$\theta_B = +4.5 \times 10^{-3}\text{ rad}$$

Using the second moment-area theorem



$$t_{B/A} = A_1(2.6\text{ m}) + A_2(0.6\text{ m})$$

$$= (-3.6\times10^{-3})(2.6\text{ m}) + (8.1\times10^{-3})(0.6\text{ m})$$

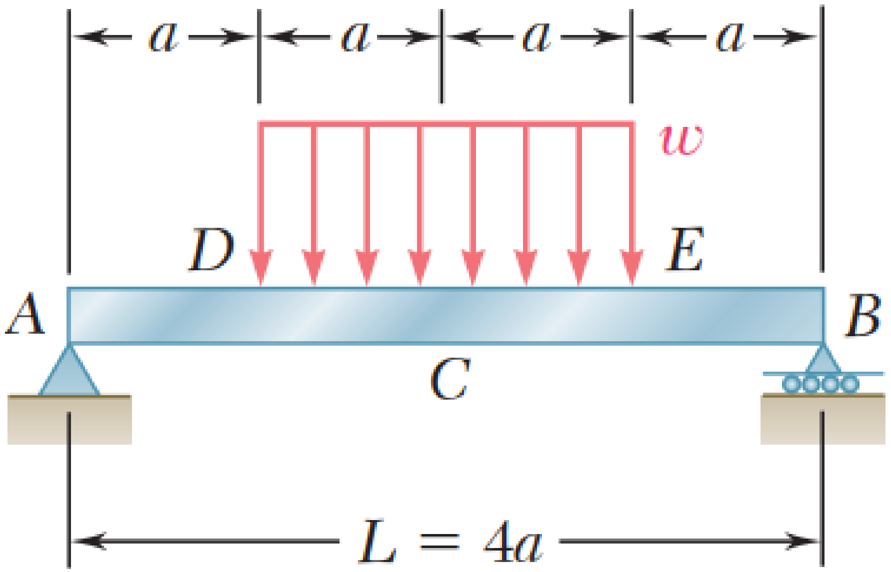
$$= -9.36\text{ mm} + 4.86\text{ mm} = -4.50\text{ mm}$$

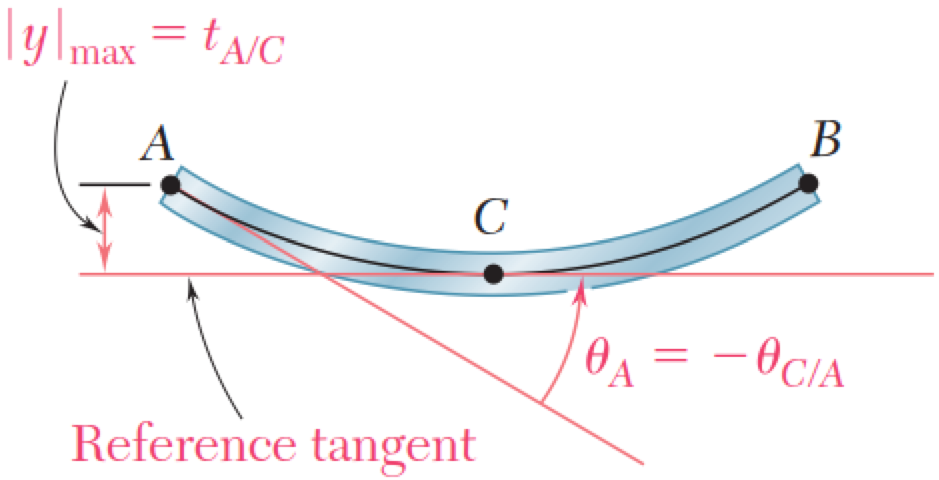
Since the reference tangent at *A* is horizontal, the deflection at *B*

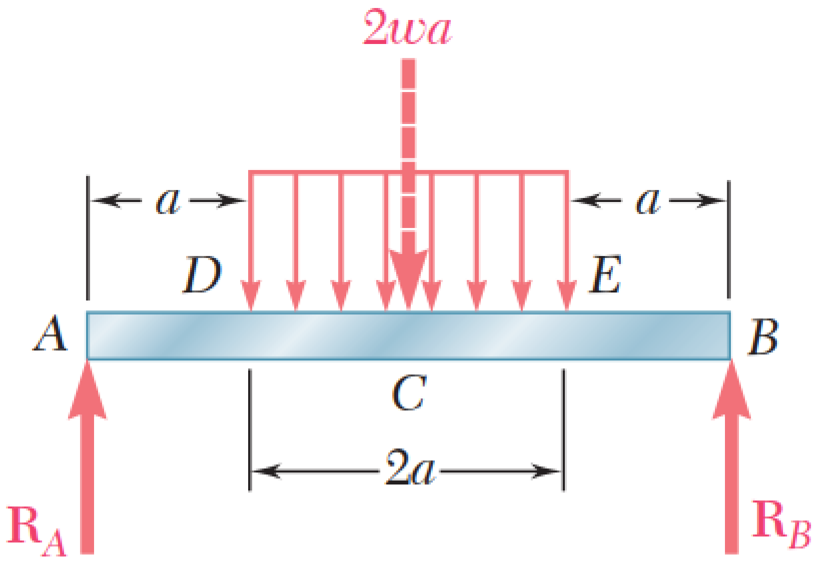
$$y_B = t_{B/A} = -4.50\text{ mm}$$

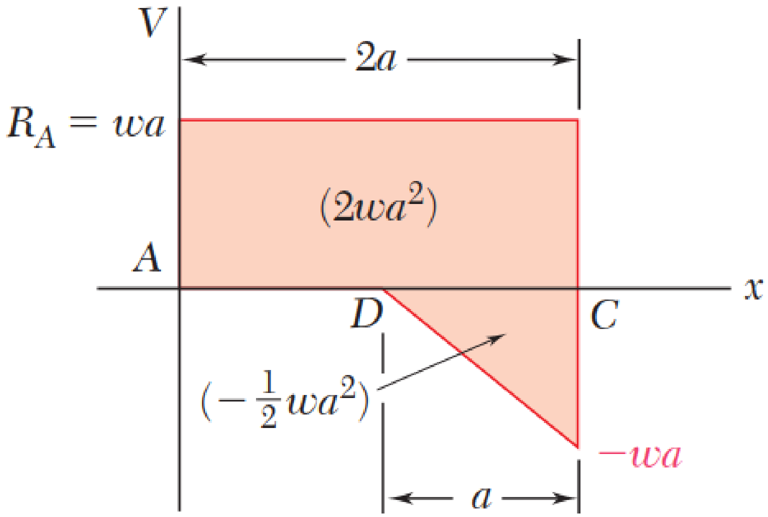
Concept Application 9.11

For the prismatic beam *AB* and the loading shown, determine the slope at a support and the maximum deflection.

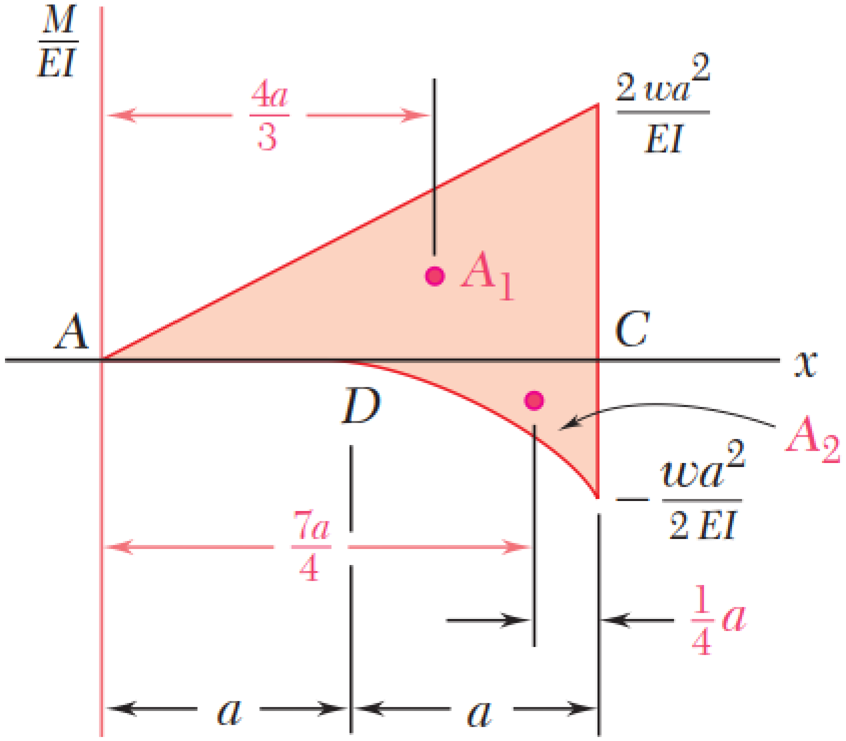








$$R_A = R_B = wa$$



$$A_1 = \frac{1}{2}(2a)\left(\frac{2wa^2}{EI}\right) = \frac{2wa^3}{EI}$$

$$A_2 = -\frac{1}{3}(a)\left(\frac{wa^2}{2EI}\right) = -\frac{wa^3}{6EI}$$

Applying the first moment-area theorem,

$$\theta_{C/A} = A_1 + A_2 = \frac{2wa^3}{EI} - \frac{wa^3}{6EI} = \frac{11wa^3}{6EI}$$

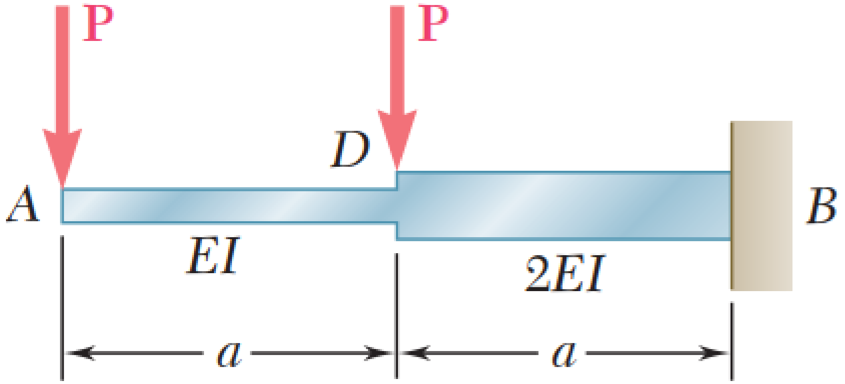
Applying the second moment-area theorem

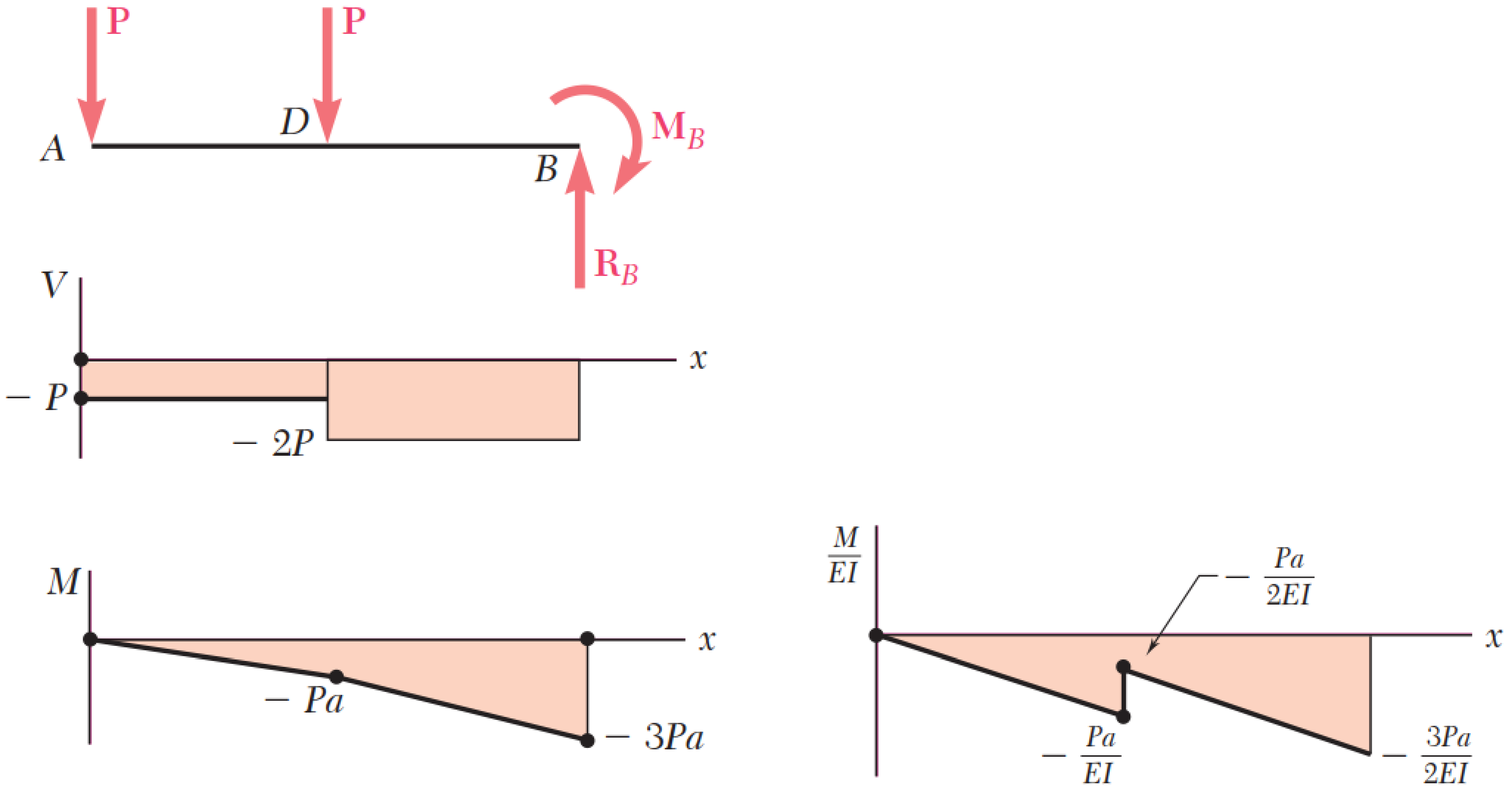
$$t_{A/C} = A_1\frac{4a}{3} + A_2\frac{7a}{4} = \left(\frac{2wa^3}{EI}\right)\frac{4a}{3} + \left(-\frac{wa^3}{6EI}\right)\frac{7a}{4} = \frac{19wa^4}{8EI}$$

$$|y|_{\max} = t_{A/C} = \frac{19wa^4}{8EI} = \frac{19wL^4}{2048EI}$$

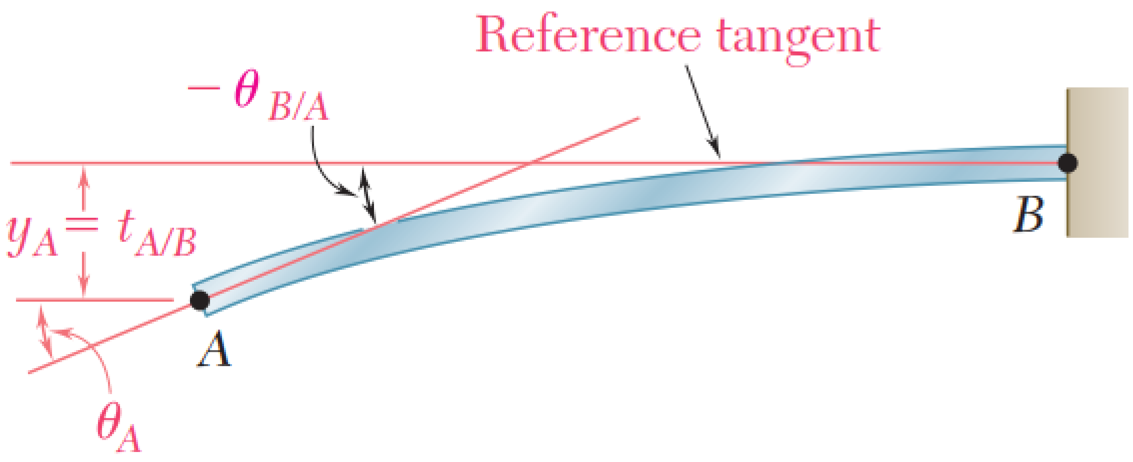
Sample Problem 9.10

Prismatic rods *AD* and *DB* are welded together to form the cantilever beam *ADB*. Knowing that the flexural rigidity is *EI* in portion *AD* of the beam and 2*EI* in portion *DB*, determine the slope and deflection at end *A* for the loading shown.



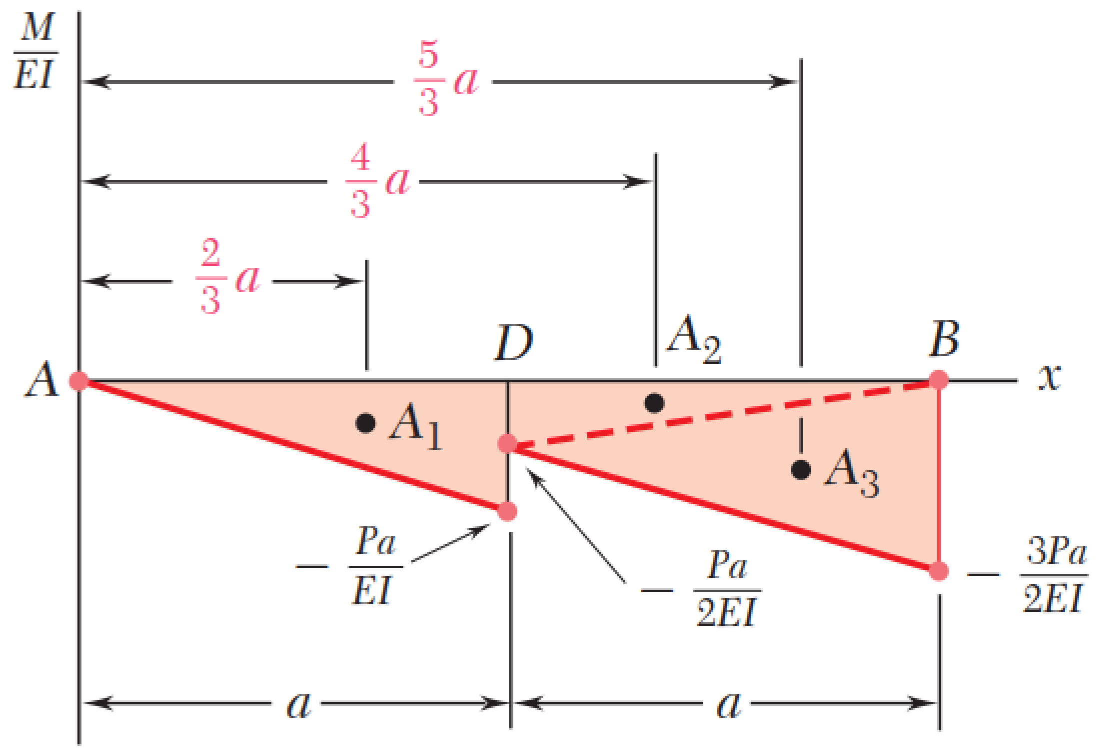


P
P
D
A
B
M_B
R_B
V
x
− P
− 2P
M
x
− Pa
− 3Pa
$\frac{M}{EI}$
$-\frac{Pa}{2EI}$
x
$-\frac{Pa}{EI}$
$-\frac{3Pa}{2EI}$



Slope and deflection at end *A* related to reference tangent at fixed end *B*.

$$\theta_A = -\theta_{B/A} \quad y_A = t_{A/B}$$



$$A_1 = -\frac{1}{2}\frac{Pa}{EI}a = -\frac{Pa^2}{2EI}$$

$$A_2 = -\frac{1}{2}\frac{Pa}{2EI}a = -\frac{Pa^2}{4EI}$$

$$A_3 = -\frac{1}{2}\frac{3Pa}{2EI}a = -\frac{3Pa^2}{4EI}$$

Using the first moment-area theorem,

$$\theta_{B/A} = A_1 + A_2 + A_3 = -\frac{Pa^2}{2EI} - \frac{Pa^2}{4EI} - \frac{3Pa^2}{4EI} = -\frac{3Pa^2}{2EI}$$

$$\theta_A = -\theta_{B/A} = +\frac{3Pa^2}{2EI} \qquad \theta_A = \frac{3Pa^2}{2EI} \measuredangle$$

Using the second moment-area theorem,

$$y_A = t_{A/B} = A_1\left(\frac{2}{3}a\right) + A_2\left(\frac{4}{3}a\right) + A_3\left(\frac{5}{3}a\right)$$

$$= \left(-\frac{Pa^2}{2EI}\right)\frac{2a}{3} + \left(-\frac{Pa^2}{4EI}\right)\frac{4a}{3} + \left(-\frac{3Pa^2}{4EI}\right)\frac{5a}{3}$$

$$y_A = -\frac{23Pa^3}{12EI} \qquad y_A = \frac{23Pa^3}{12EI}\downarrow$$