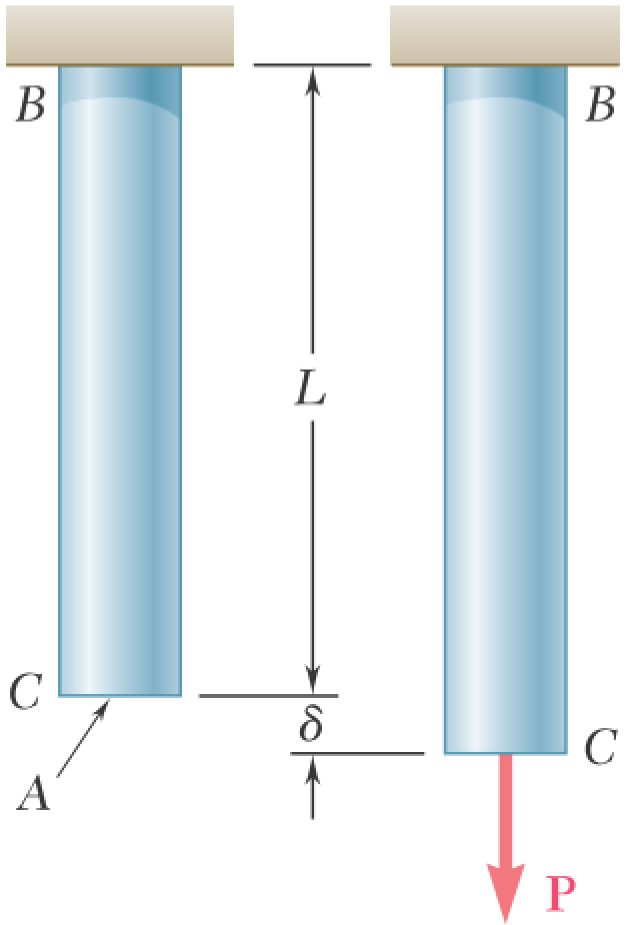
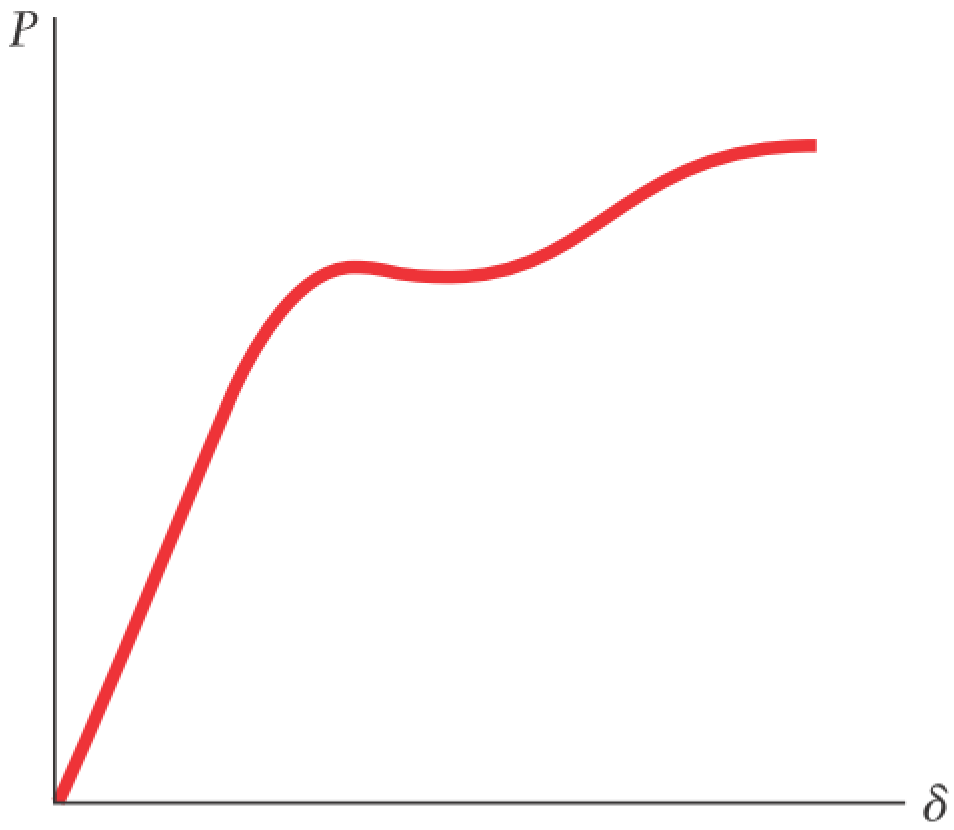
CH.2 STRESS AND STRAIN
AXIAL LOADING

Part I

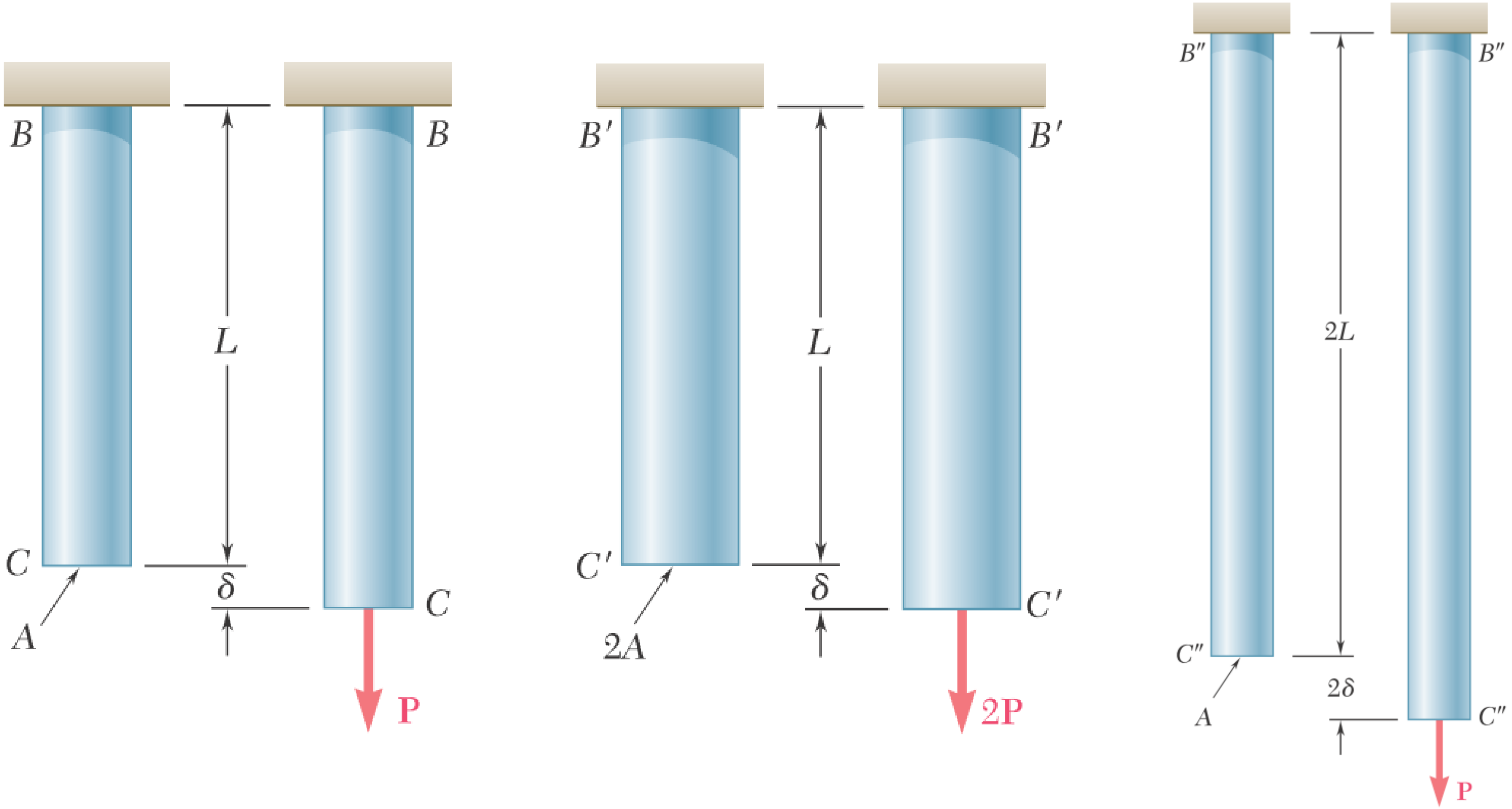
Objectives

- Concept of strain.
- Relationship between stress and strain in different materials.
- Determine the deformation of structural components under axial loading.
- Hooke’s Law and the modulus of elasticity.
- Concept of lateral strain and Poisson's ratio.
- Solve indeterminate problems using axial deformations.
- Saint-Venant’s principle and the distribution of stresses.





- From the load-displacement diagrams we can obtain useful informations to the analysis of the rod for the BC rod. But, can it be used to predict the deformation of a rod of the same material but with different dimensions?



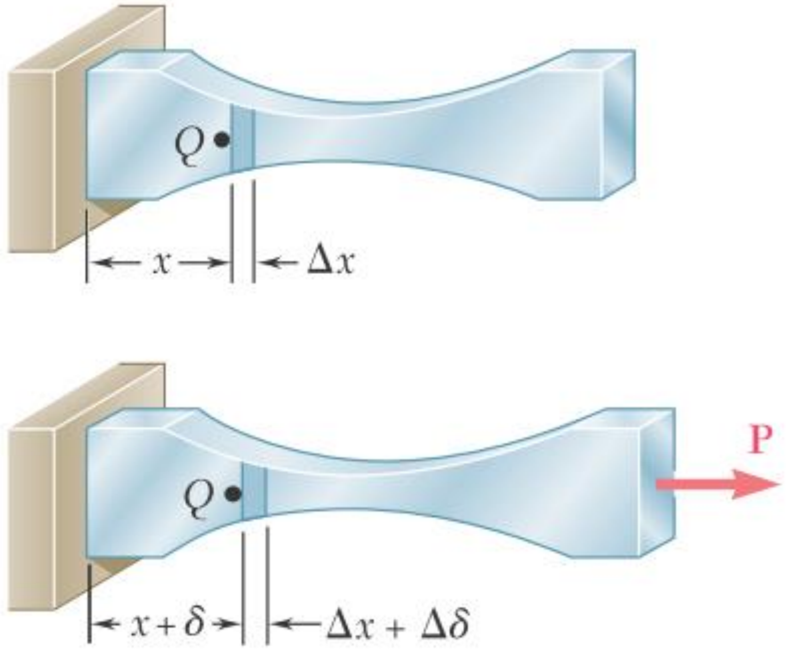
- In both cases, the ratio of the deformation over the length of the rod is the same at $\delta/_{L}$.

- Here, we have a new term. Deformation per unit length in a rod under axial loading is called as ***normal strain***.

- The **normal strain**, ε (Greek letter *epsilon*), is $$\varepsilon = \frac{\delta}{L}$$

- Since deformation and length are expressed in the same units, the normal strain ε obtained by dividing δ by L is a *dimensionless quantity*.

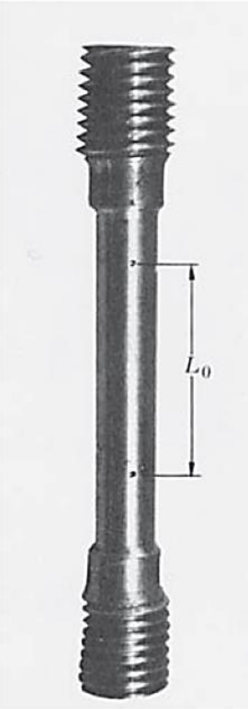
- Plotting the stress $\sigma = \frac{P}{A}$ against the strain $\varepsilon = \frac{\delta}{L}$ results in a curve that is characteristic of the properties of the material but does not depend upon the dimensions of the specimen used. This curve is called a **stress-strain diagram**.

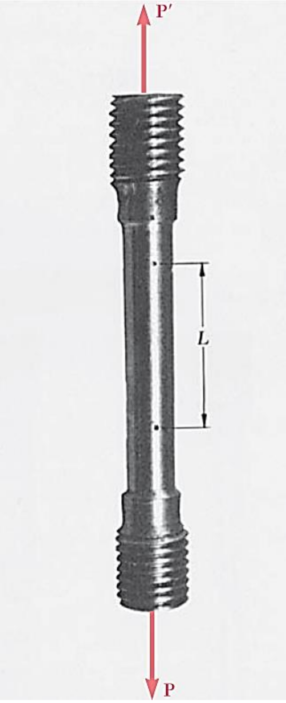


- If the rod has a uniform cross section of area A, the normal stress σ is assumed to have a constant value P/A throughout the rod. The strain ε is the ratio of the total deformation δ over the total length L of the rod.
- If the rod has a variable cross-sectional area A, the normal stress $\sigma = P/A$ varies along the member, and it is necessary to define the strain at a given point Q by considering a small element of undeformed length Δx. Denoting the deformation of the element under the given loading by $\Delta\delta$, the normal strain at point Q is defined as

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx}$$

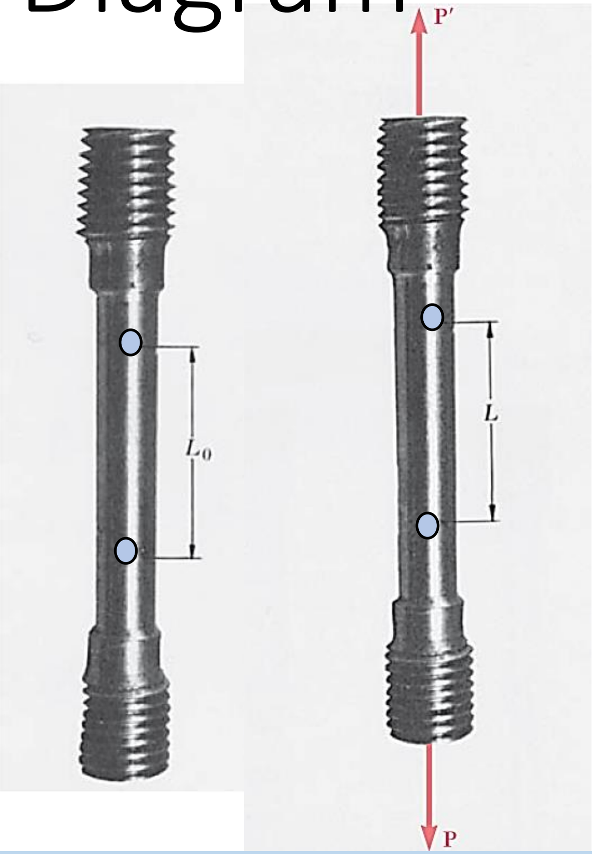
Stress-Strain Diagram





- To obtain the stress-strain diagram of a material, a **tensile test** is conducted on a specimen of the material.
- The cross-sectional area of the cylindrical central portion of the specimen is accurately determined and two gage marks are inscribed on that portion at a distance L_0 from each other.
- The distance L_0 is known as the *gage length* of the specimen.

Stress-Strain Diagram



Tensile Testing Procedure

- The test specimen is then placed in a testing machine, which is used to apply a centric load *P*. As load *P* increases, the distance *L* between the two gage marks also increases. The distance *L* is measured with a dial gage, and the elongation $\delta = L - L_0$ is recorded for each value of *P*. A second dial gage is often used simultaneously to measure and record the change in diameter of the specimen.

From each pair of readings P and δ, the **engineering stress** σ is

$$\sigma = \frac{P}{A_0}$$

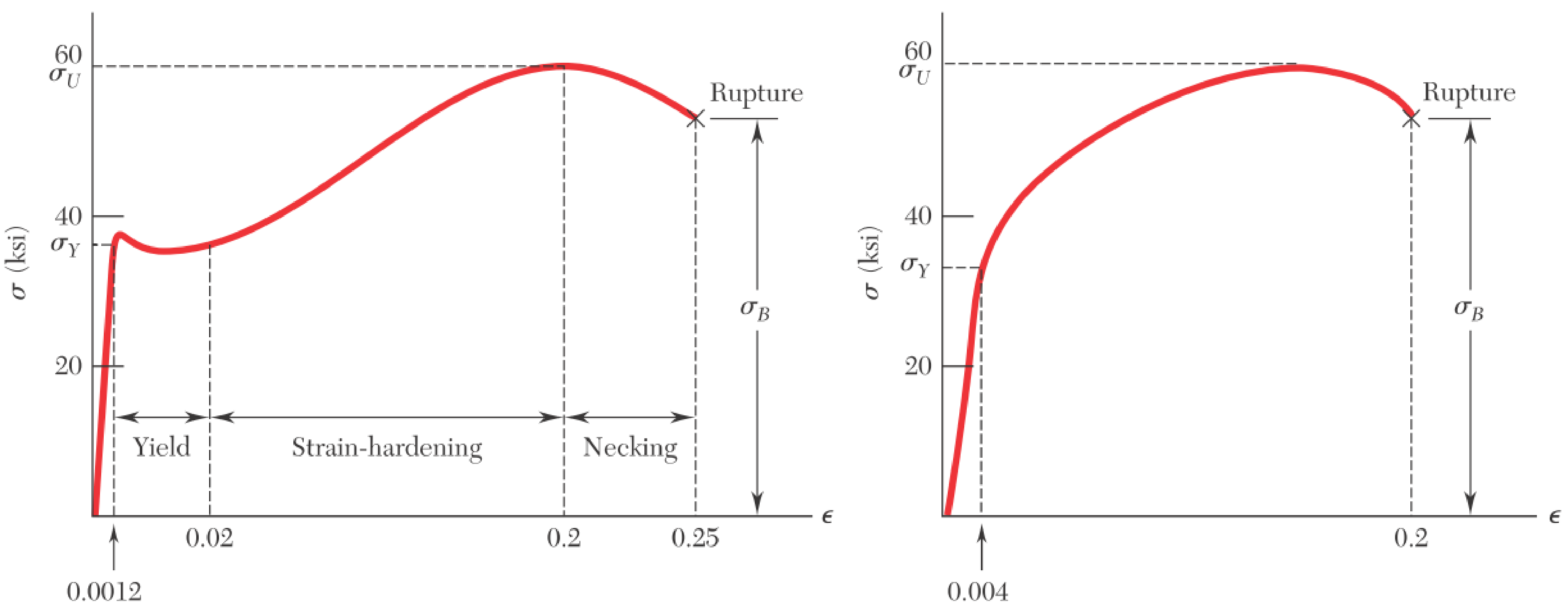
and the **engineering strain** ε is

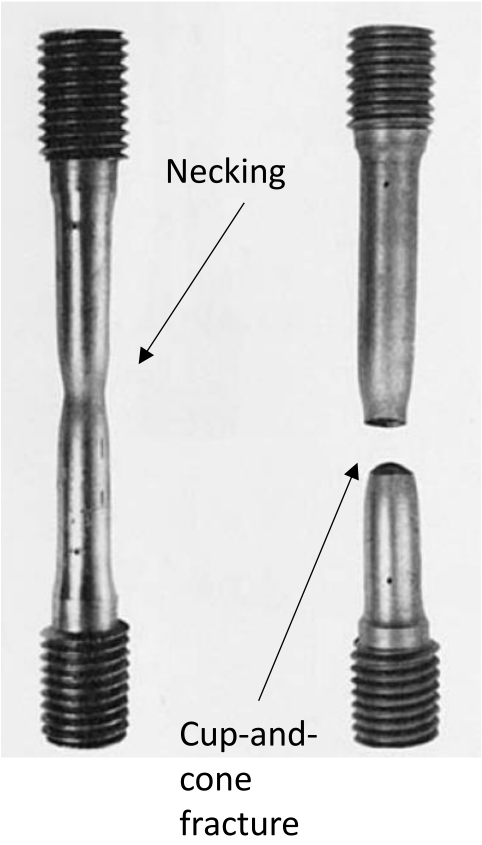
$$\varepsilon = \frac{\delta}{L_0}$$

- The stress-strain diagram can be obtained by plotting ε as an abscissa and σ as an ordinate.

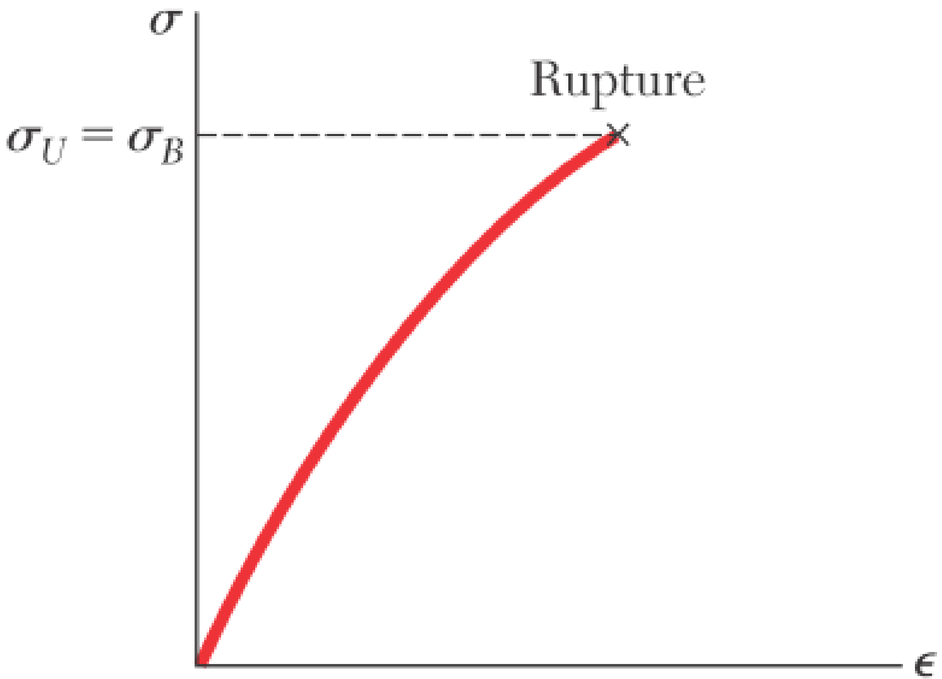
- Stress-strain diagrams of materials vary widely, and different tensile tests conducted on the same material may yield different results, depending upon the temperature of the specimen and the speed of loading. However, some common characteristics can be distinguished from stress-strain diagrams to divide materials into two broad categories: **ductile** and **brittle materials**.

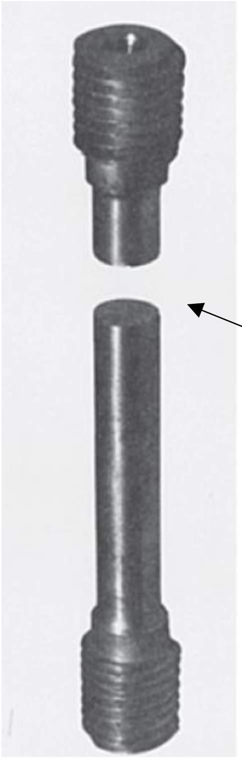
Stress-strain diagrams of two typical ductile materials





Stress-strain diagram of a typical brittle material





- A standard measure of the ductility of a material is its ***percent elongation***:

$$Percent\ elongation = 100\frac{L_B - L_0}{L_0}$$

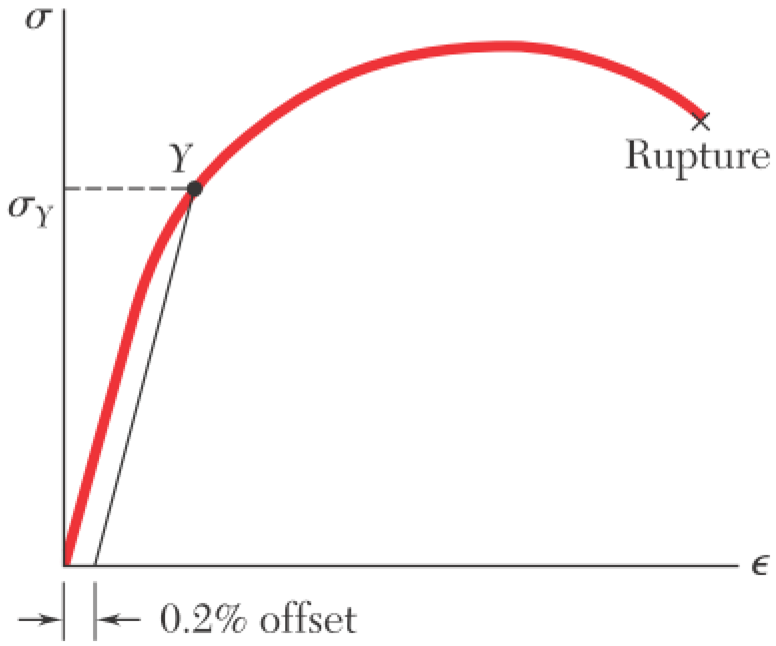
where L_0 and L_B are the initial length of the tensile test specimen and its final length at rupture, respectively.

- Another measure of ductility that is sometimes used is the ***percent reduction in area***:

$$Percent\ reduction\ in\ area = 100\frac{A_0 - A_B}{A_0}$$

where A_0 and A_B are the initial cross-sectional area of the specimen and its minimum cross-sectional area at rupture, respectively.

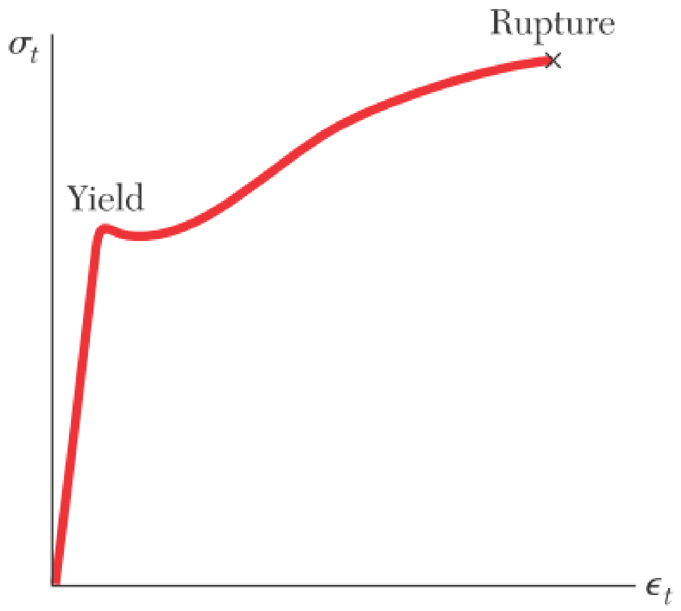
Determination of Yield Strength



The **yield strength (σ_y)** is the stress at which **permanent deformation** begins.

True Stress and True Strain

- Engineering stress $\sigma = \dfrac{P}{A_0}$
- True stress $\sigma_t = \dfrac{P}{A}$



- The difference between the engineering stress σ and the true stress σ_t becomes apparent in ductile materials after yield has started.
- While the engineering stress σ, which is directly proportional to the load P, decreases with P during the necking phase, the true stress σ_t, which is proportional to P but also inversely proportional to A (instantaneous area), keeps increasing until rupture of the specimen occurs.

- For engineering strain $\varepsilon = \delta/L_0$, instead of using the total elongation δ and the original value L_0 of the gage length, many scientists use all of the values of L that they have recorded. Dividing each increment ΔL of the distance between the gage marks by the corresponding value of L, the elementary strain $\Delta\varepsilon = \Delta L/L$. Adding the successive values of $\Delta\varepsilon$, the true strain ε_t is

$$\varepsilon_t = \sum \Delta\varepsilon = \sum (\Delta L/L)$$

- With the summation replaced by an integral, the true strain can be expressed as:

$$\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0}$$

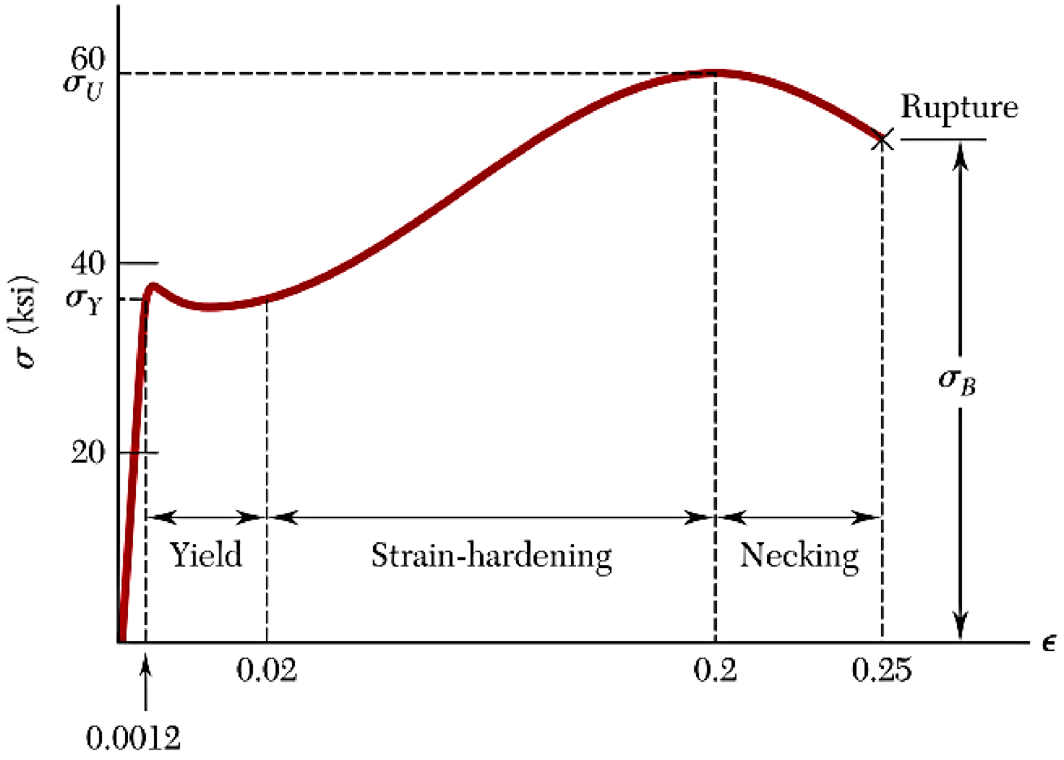
Hooke's Law; Modulus of Elasticity

- Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion (elastic region) of the corresponding stress-strain diagram. For that initial portion of the diagram, the stress σ is directly proportional to the strain ε:

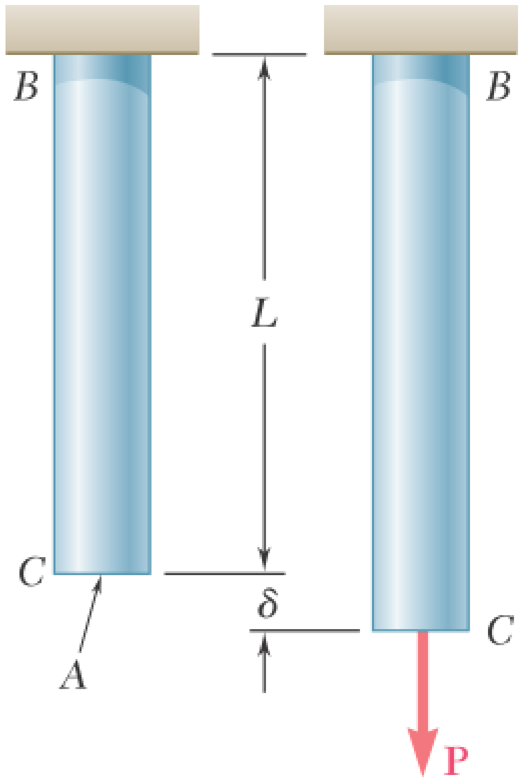
$$\sigma = E\varepsilon$$

- This is known as **Hooke's law.**

- The coefficient E of the material is the **modulus of elasticity** or **Young's modulus**.



Deformations of Members Under Axial Loading



- If the resulting axial stress $\sigma = P/A$ does not exceed the proportional limit of the material, Hooke's law applies and

$$\sigma = E\varepsilon$$

- from which

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- Recalling that the strain $\varepsilon = \frac{\delta}{L}$ and substituting $\varepsilon = \frac{P}{AE}$ into $\delta = \varepsilon L$ gives

$$\boxed{\boldsymbol{\delta = \frac{PL}{AE}}}$$

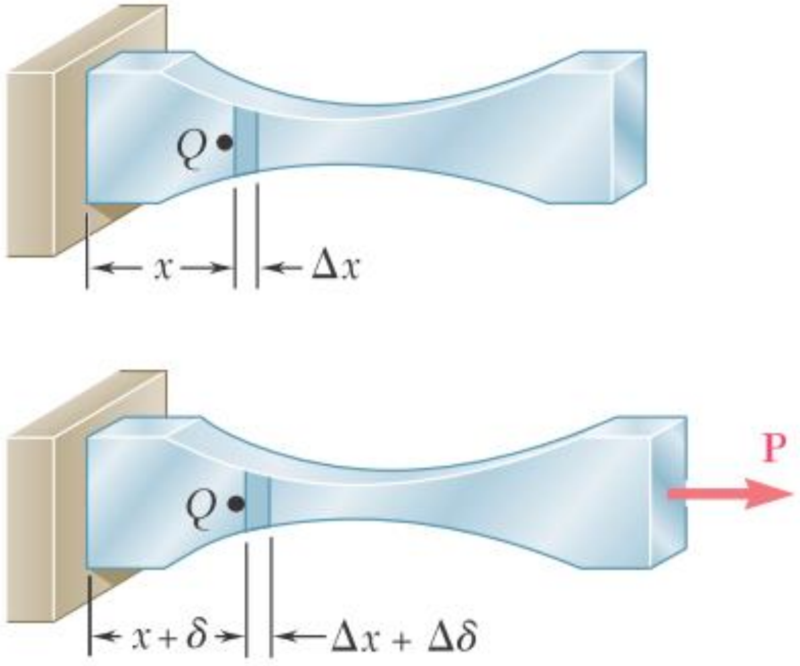
Deformation of Members Under Axial Loading

- Equation $\delta = \frac{PL}{AE}$ can be used only if the rod is homogeneous (constant E), has a uniform cross section of area A, and is loaded at its ends.
- If the rod is loaded at other points, or consists of several portions of various cross sections and possibly of different materials, it must be divided into component parts that satisfy the required conditions.
- Using the internal force P_i, length L_i, cross-sectional area A_i, and modulus of elasticity E_i, corresponding to part i, the deformation of the entire rod is

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Deformation of Graded Members Under Axial Loading

- In the case of a member of variable cross section, the strain ε depends upon the position of the point Q, where it is computed as $\varepsilon = d\delta/dx$. Solving for $d\delta$ and substituting for ε from $\varepsilon = P/AE$, the deformation of an element of length dx is



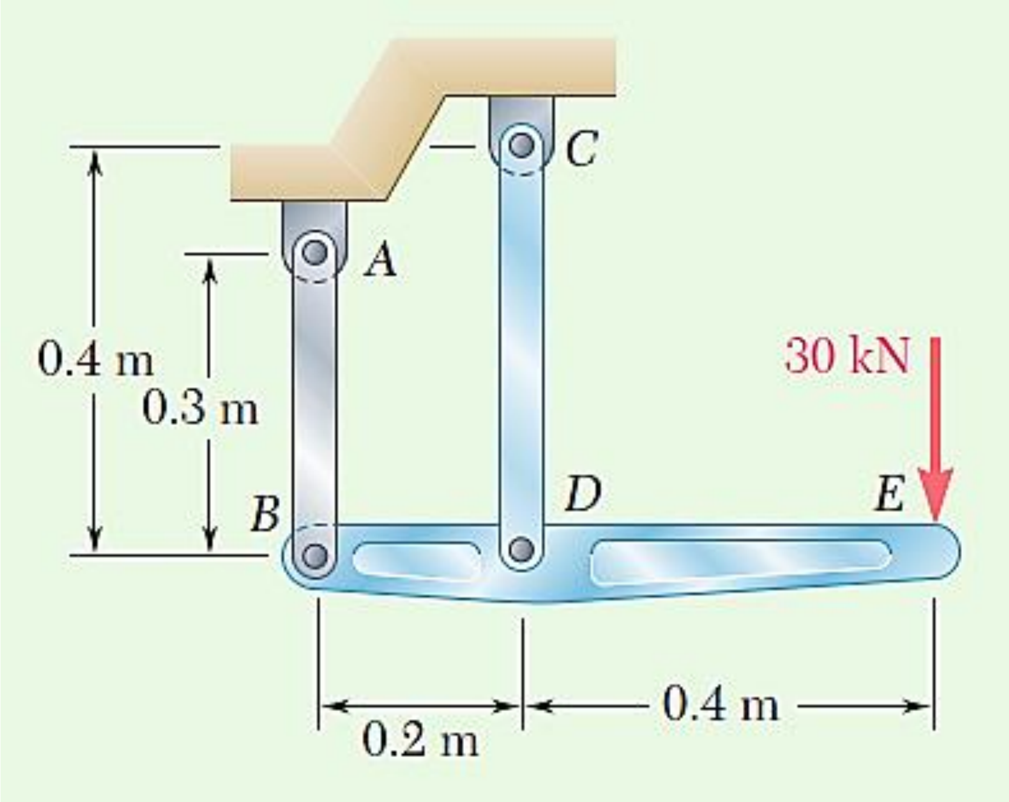
$$d\delta = \varepsilon dx = \frac{Pdx}{AE}$$

The total deformation δ of the member is obtained by integrating this expression over the length L of the member:

$$\delta = \int_0^L \frac{Pdx}{AE}$$

Example 2.1

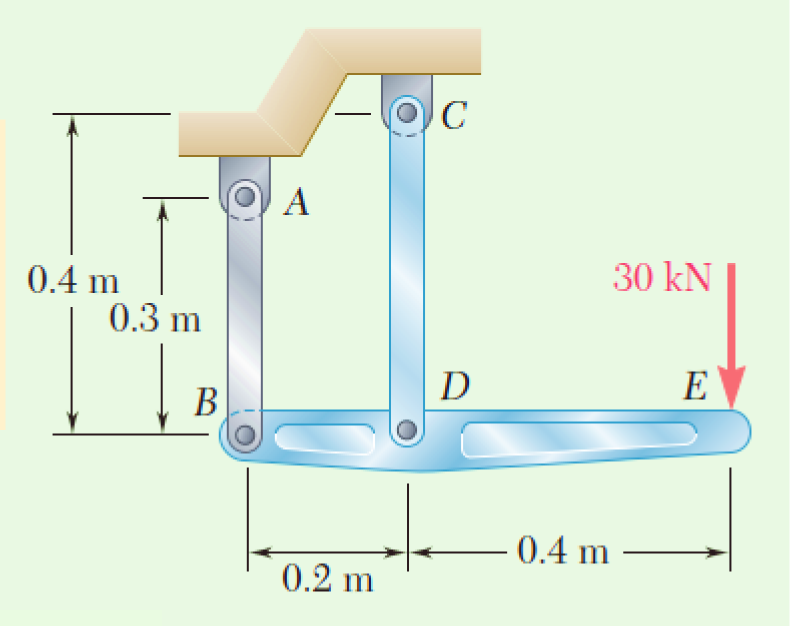
The rigid bar BDE is supported by two links AB and CD. Link AB is made of aluminum (E=70 GPa) and has a cross-sectional area of 500 mm^2. Link CD is made of steel (E= 200 GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (a) of B, (b) of D, and (c) of E.



Example 2.1

STRATEGY: Consider the free body of the rigid bar to determine the internal force of each link. Knowing these forces and the properties of the links, their deformations can be evaluated. You can then use simple geometry to determine the deflection of E.

MODELING: Draw the free body diagrams of the rigid bar and the two links



ANALYSIS:

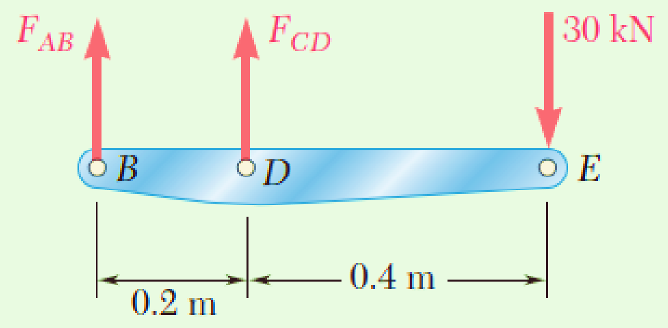


Fig. 1 Free-body diagram of rigid bar *BDE*.

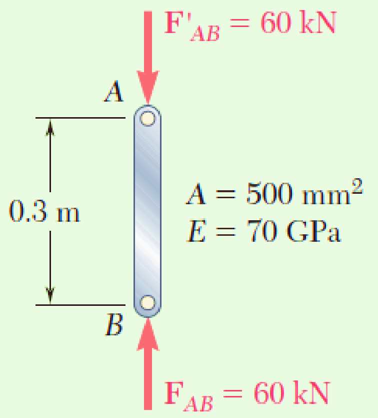


Fig. 2 Free-body diagram of two-force member *AB*.

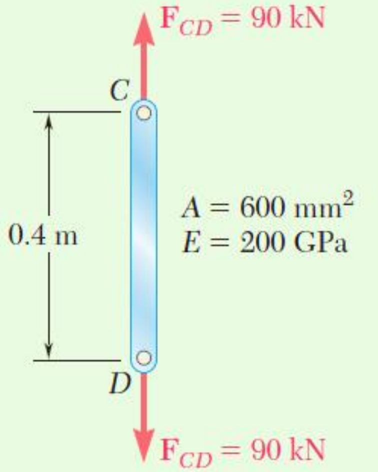
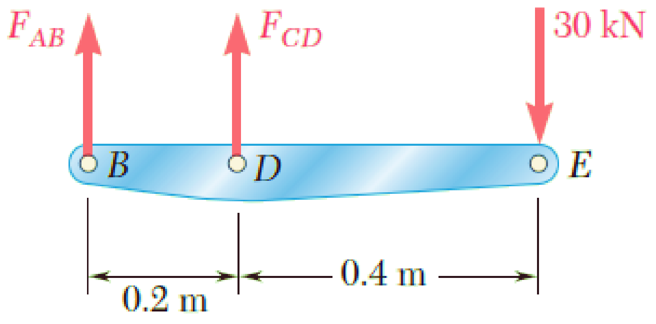


Fig. 3 Free-body diagram of two-force member *CD*.

Example 2.1

ANALYSIS:



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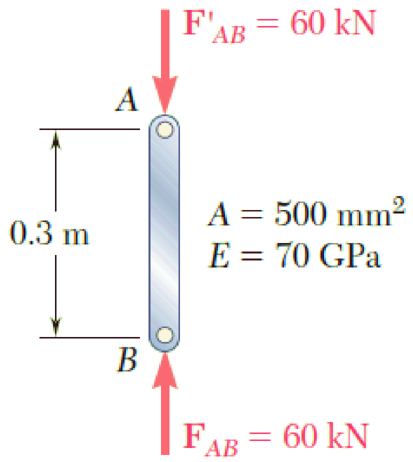

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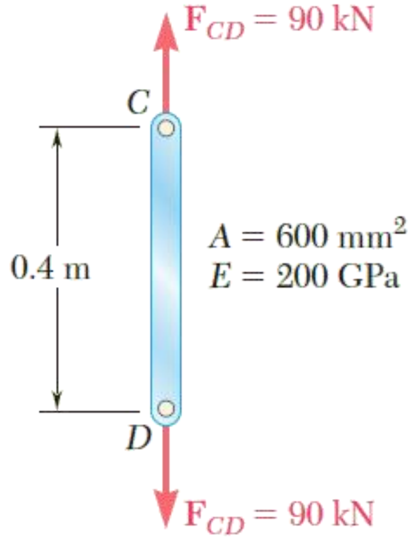
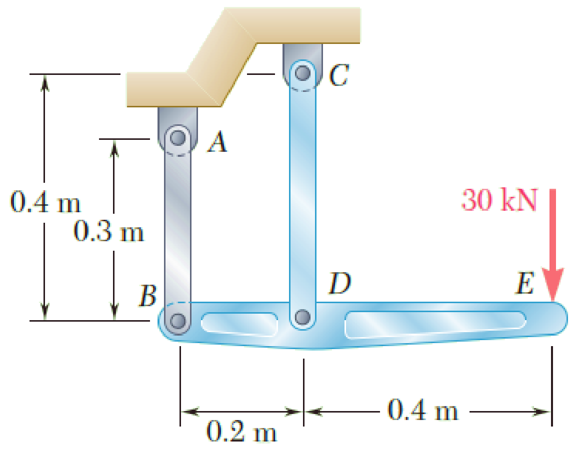


Fig. 3 Free-body diagram of two-force member *CD*.



Free Body: Bar BDE (Fig. 1)

$$+\circlearrowleft \Sigma M_B = 0: \quad -(30\text{ kN})(0.6\text{ m}) + F_{CD}(0.2\text{ m}) = 0$$

$$F_{CD} = +90\text{ kN} \qquad F_{CD} = 90\text{ kN} \quad \textit{tension}$$

$$+\circlearrowleft \Sigma M_D = 0: \quad -(30\text{ kN})(0.4\text{ m}) - F_{AB}(0.2\text{ m}) = 0$$

$$F_{AB} = -60\text{ kN} \qquad F_{AB} = 60\text{ kN} \quad \textit{compression}$$

a. Deflection of B. Since the internal force in link AB is compressive (Fig. 2), P = -60 kN and

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3\text{ N})(0.3\text{ m})}{(500 \times 10^{-6}\text{ m}^2)(70 \times 10^9\text{ Pa})} = -514 \times 10^{-6}\text{ m}$$

The negative sign indicates a contraction of member AB. Thus, the deflection of end B is upward:

$$\delta_B = \mathbf{0.514\text{ mm}} \uparrow$$

Example 2.1

ANALYSIS:

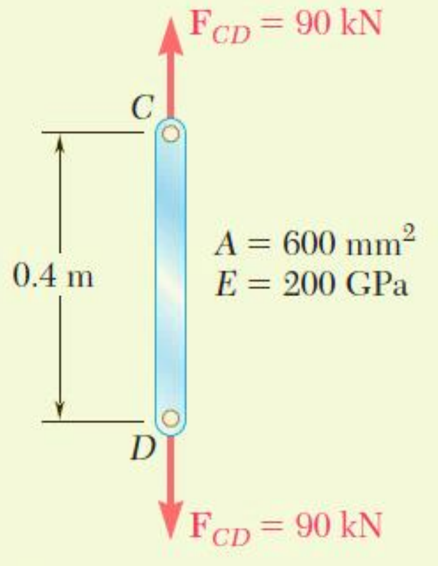


Fig. 3 Free-body diagram of two-force member *CD*.

b. Deflection of D. Since in rod CD (Fig. 3), P = 90 kN, write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3\,\text{N})(0.4\,\text{m})}{(600 \times 10^{-6}\,\text{m}^2)(200 \times 10^9\,\text{Pa})} = 300 \times 10^{-6}\,\text{m} \qquad \delta_D = \mathbf{0.300\ mm} \downarrow$$

c. Deflection of E. Referring to Fig. 4, we denote by B' and D' the displaced positions of points B and D. Since the bar BDE is rigid, points B', D', and E' lie in a straight line. Therefore,

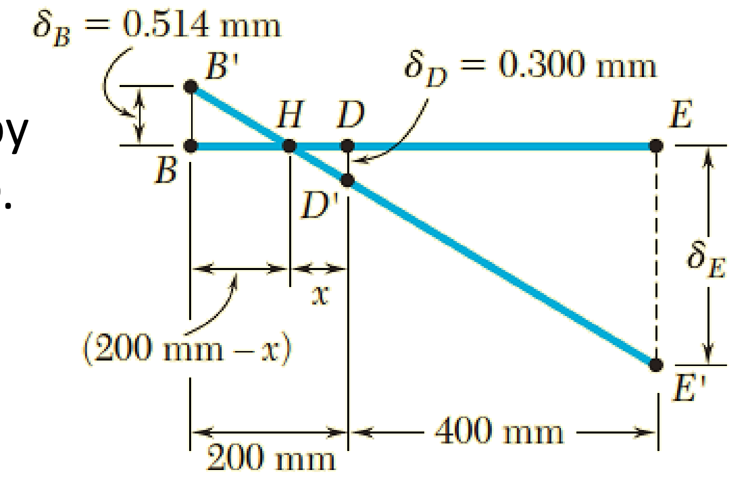


Fig. 4 Deflections at *B* and *D* of rigid bar are used to find δ_E.

$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514\,\text{mm}}{0.300\,\text{mm}} = \frac{(200\,\text{mm}) - x}{x} \qquad x = 73.7\,\text{mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300\,\text{mm}} = \frac{(400\,\text{mm}) + (73.7\,\text{mm})}{73.7\,\text{mm}}$$

$$\delta_E = \mathbf{1.928\ mm} \downarrow$$