Ch.4 PURE BENDING

Part I

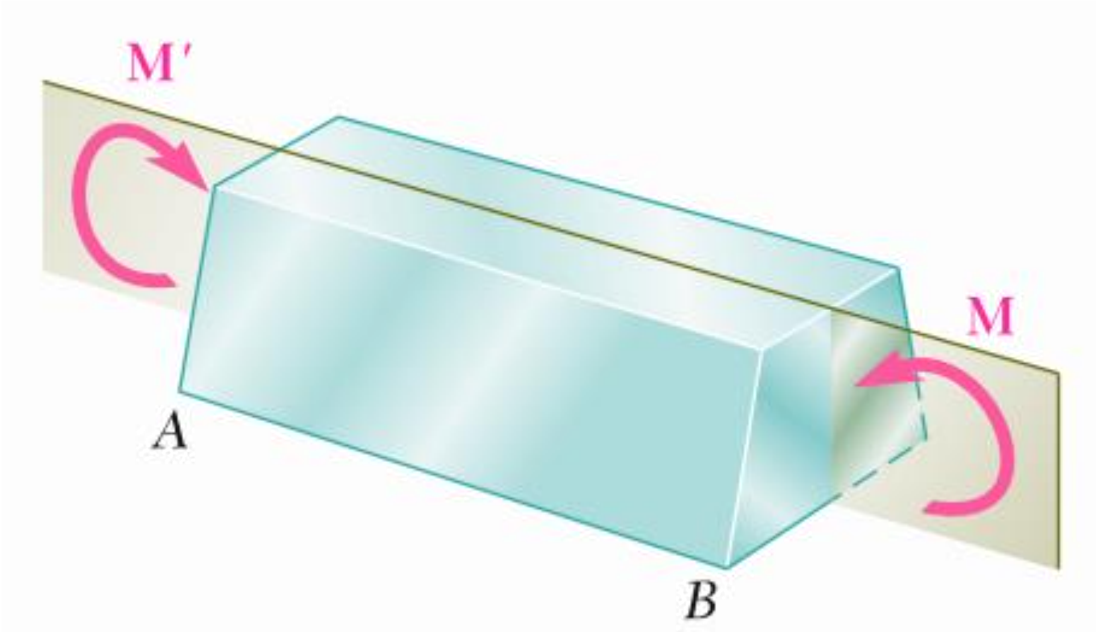
Objectives

- Understand the bending behavior
- Define the deformations, strains, and normal stresses in beams subject to pure bending
- Describe the behavior of composite beams made of more than one material
- Analyze members subject to eccentric axial loading, involving both axial stresses and bending stresses
- Review beams subject to unsymmetric bending, i.e., where bending does not occur in a plane of symmetry
- Study bending of curved members

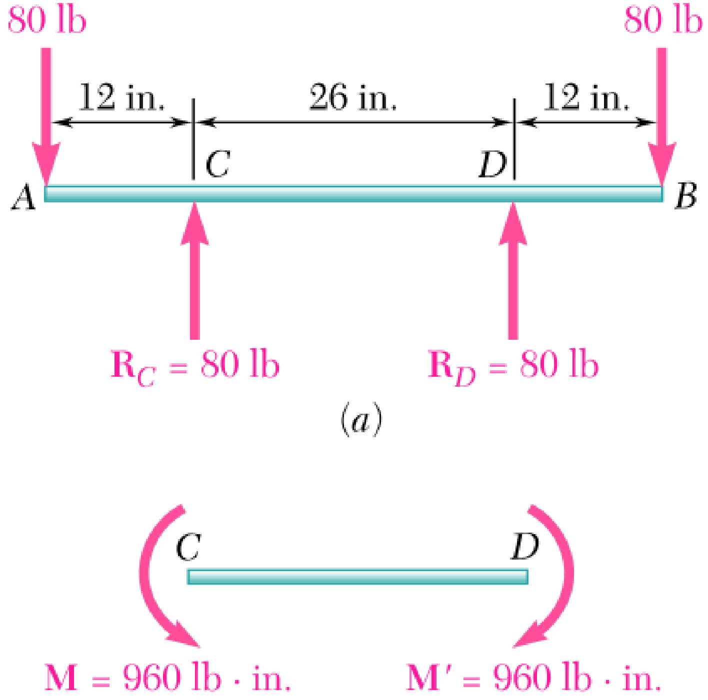
INTRODUCTION

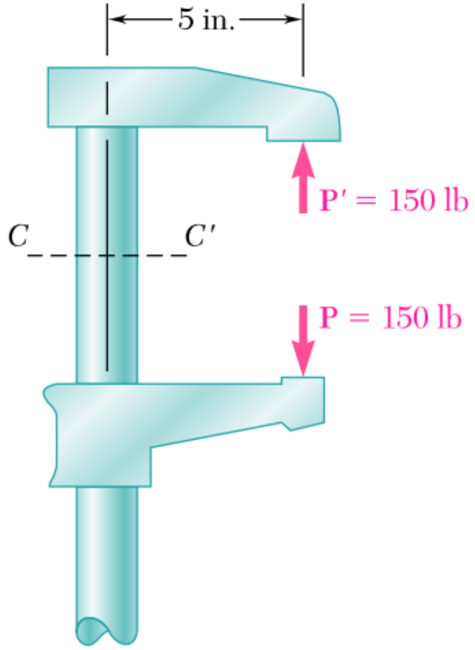
Bending is a major concept used in the design of many machine and structural components, such as beams and girders.

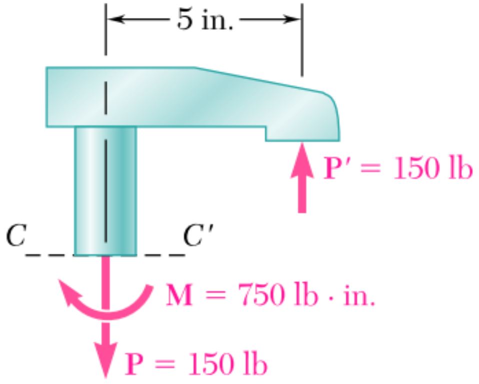
This chapter is devoted to the analysis of prismatic members subjected to equal and opposite couples M and M' acting in the same longitudinal plane. Such members are said to be in pure bending. The members are assumed to possess a plane of symmetry with the couples M and M' acting in that plane.



- An example of pure bending is provided by the bar of a typical barbell as it is held overhead by a weightlifter as shown in the photo.
- The bar carries equal weights at equal distances from the hands of the weightlifter. Because of the symmetry of the free-body diagram of the bar, the reactions at the hands must be equal and opposite to the weights.
- Therefore, as far as the middle portion CD of the bar is concerned, the weights and the reactions can be replaced by two equal and opposite 960-lb·in. couples, showing that the middle portion of the bar is in pure bending.



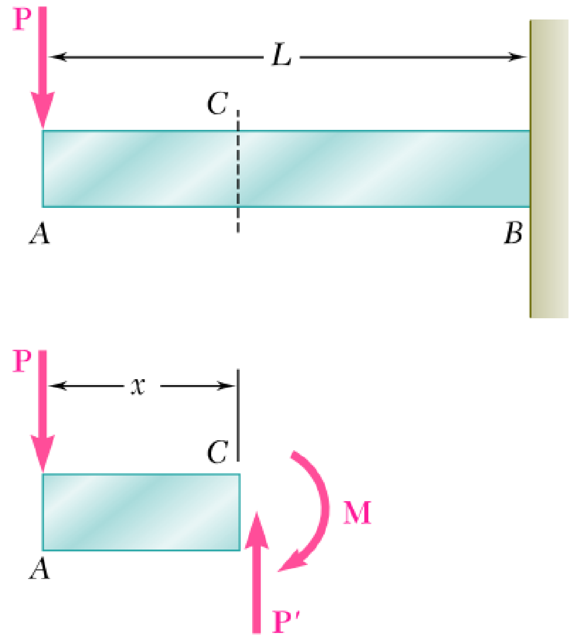




- Photo shows a 12-in. steel bar clamp used to exert 150-lb forces on two pieces of lumber as they are being glued together. Figure (left) shows the equal and opposite forces exerted by the lumber on the clamp. These forces result in an eccentric loading of the straight portion of the clamp. In Figure (right), a section CC’ has been passed through the clamp and a free-body diagram has been drawn of the upper half of the clamp. The internal forces in the section are equivalent to a 150-lb axial tensile force P and a 750-lb·in. couple M. By combining our knowledge of the stresses under a centric load and the results of an analysis of stresses in pure bending, the distribution of stresses under an eccentric load is obtained.

- The study of pure bending plays an essential role in the study of beams (i.e., prismatic members) subjected to various types of transverse loads.

- Consider a cantilever beam AB supporting a concentrated load P at its free end. If a section is passed through C at a distance x from A, the free-body diagram of AC shows that the internal forces in the section consist of a force P' equal and opposite to P and a couple M of magnitude $M = Px$. The distribution of normal stresses in the section can be obtained from the couple M as if the beam were in pure bending.

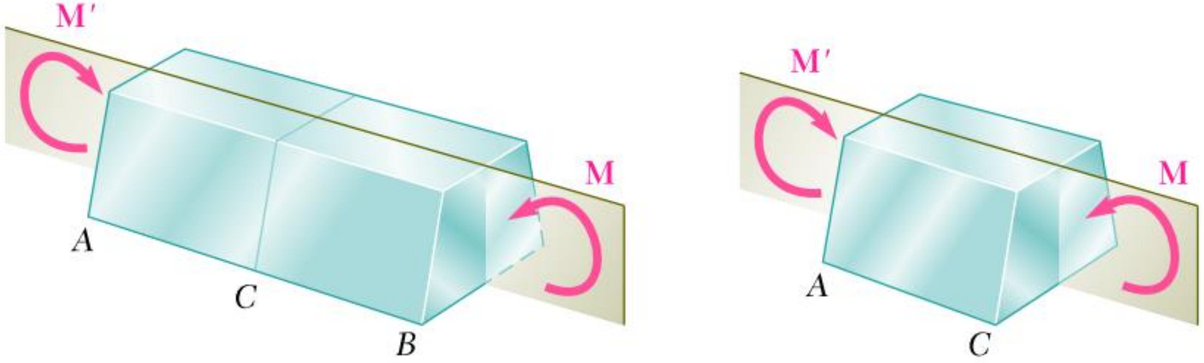


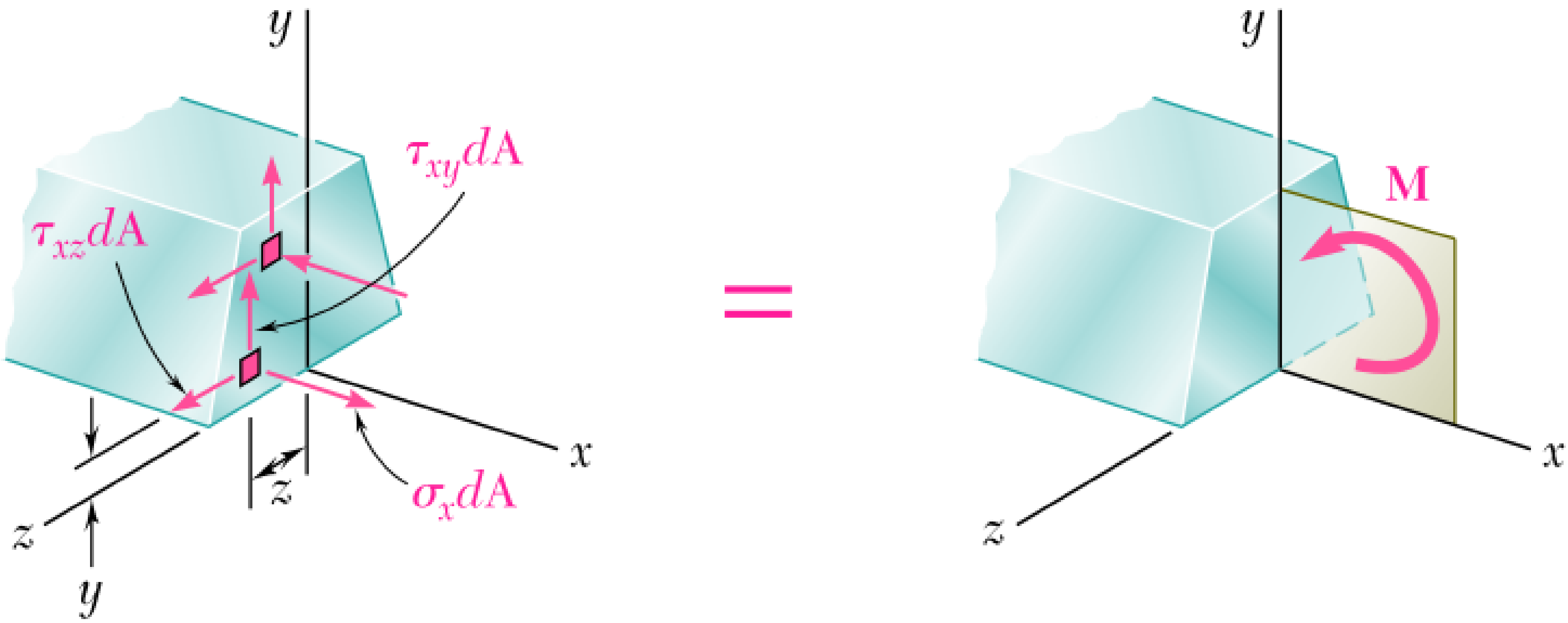
- The shearing stresses in the section depend on the force P', and their distribution over a given section is discussed in the next chapter.

Symmetric Members In Pure Bending

Internal Moment and Stress Relations

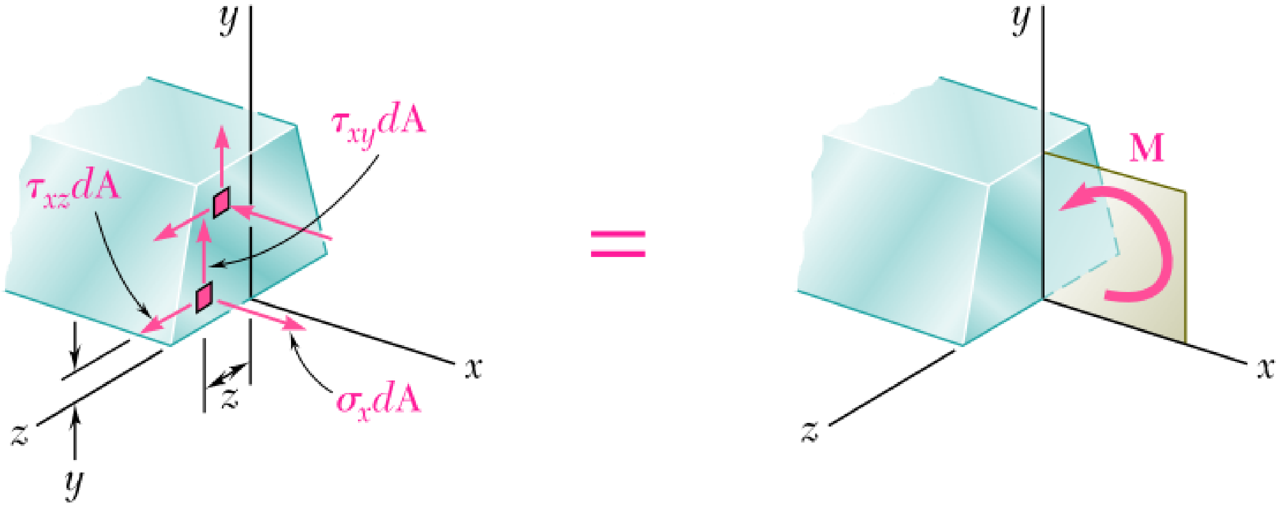
- Consider a prismatic member AB possessing a plane of symmetry and subjected to equal and opposite couples M and M' acting in that plane. If a section is passed through the member AB at some arbitrary point C, the conditions of equilibrium of the portion AC of the member require the internal forces in the section to be equivalent to the couple **M**. The moment **M** of that couple is the bending moment in the section.





- Denoting by σ_x the normal stress at a given point of the cross section and by τ_{xy} and τ_{xz} the components of the shearing stress, we express that the system of the elementary internal forces exerted on the section is equivalent to the couple M.

- Recall from statics that a couple **M** actually consists of two equal and opposite forces. The sum of the components of these forces in any direction is therefore equal to zero. Moreover, the moment of the couple is the same about any axis perpendicular to its plane and is zero about any axis contained in that plane. Selecting arbitrarily the z axis shown in figure, the equivalence of the elementary internal forces and the couple **M** is expressed by writing that the sums of the components and moments of the forces are equal to the corresponding components and moments of the couple **M**:

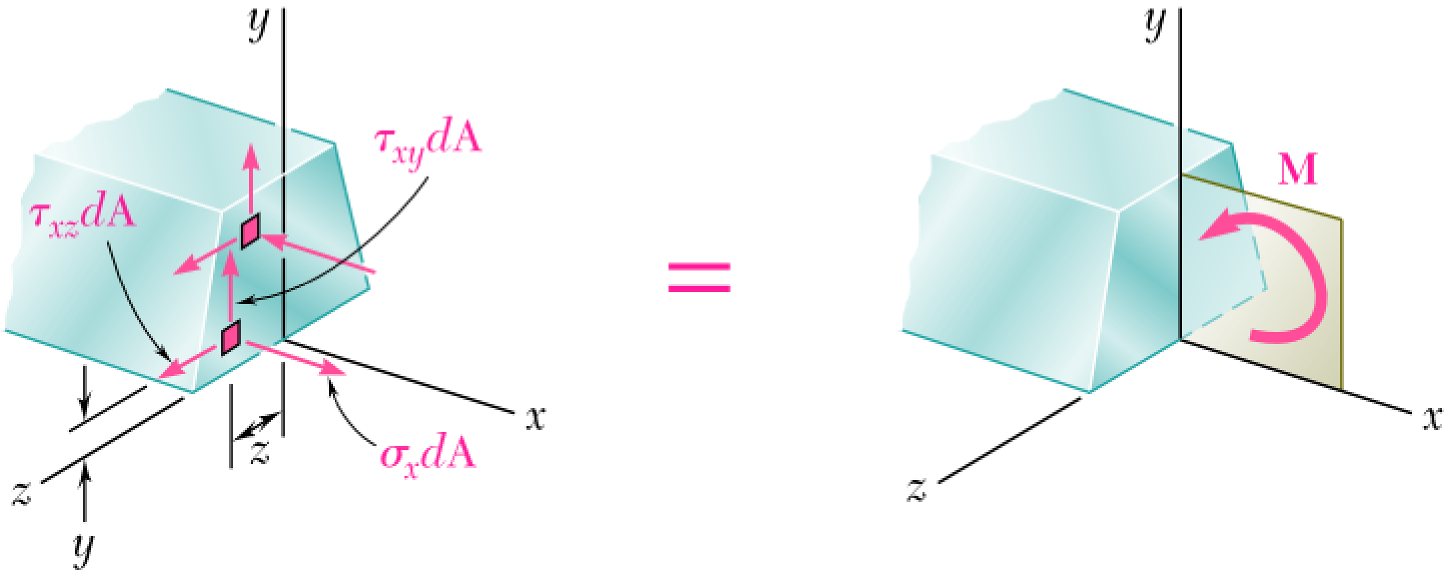


x components: $\int \sigma_x \, dA = 0$

Moments about y axis: $\int z\sigma_x \, dA = 0$

Moments about z axis: $\int (-y\sigma_x \, dA) = M$

- Three additional equations could be obtained by setting equal to zero the sums of the y components, z components, and moments about the x axis, but these equations would involve only the components of the shearing stress and, as you will see in the next section, the components of the shearing stress are both equal to zero.

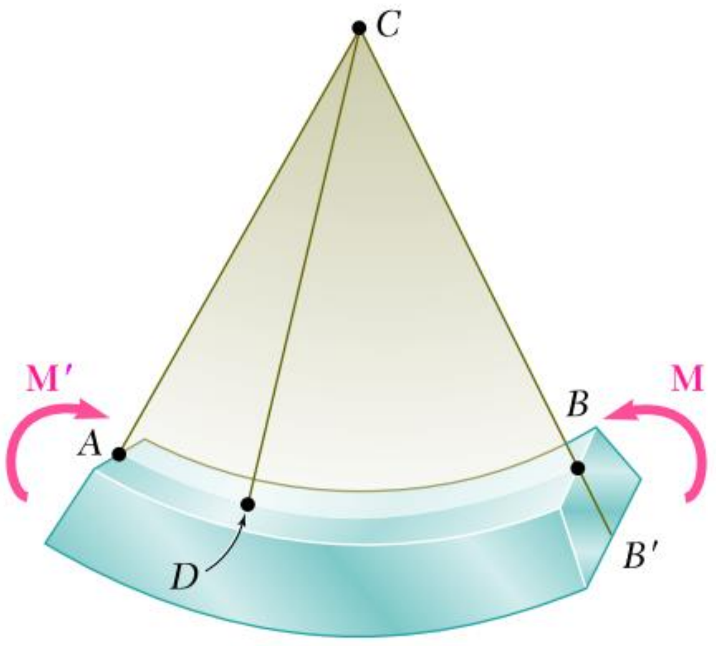


- Two remarks should be made at this point:
  - The minus sign in $\int(-y\sigma_x dA) = M$ is due to the fact that a tensile stress ($\sigma_x > 0$) leads to a negative moment (clockwise) of the normal force $\sigma_x dA$ about the z axis.
  - $\int z\sigma_x dA = 0$ could have been anticipated, since the application of couples in the plane of symmetry of member AB result in a distribution of normal stresses symmetric about the y axis.

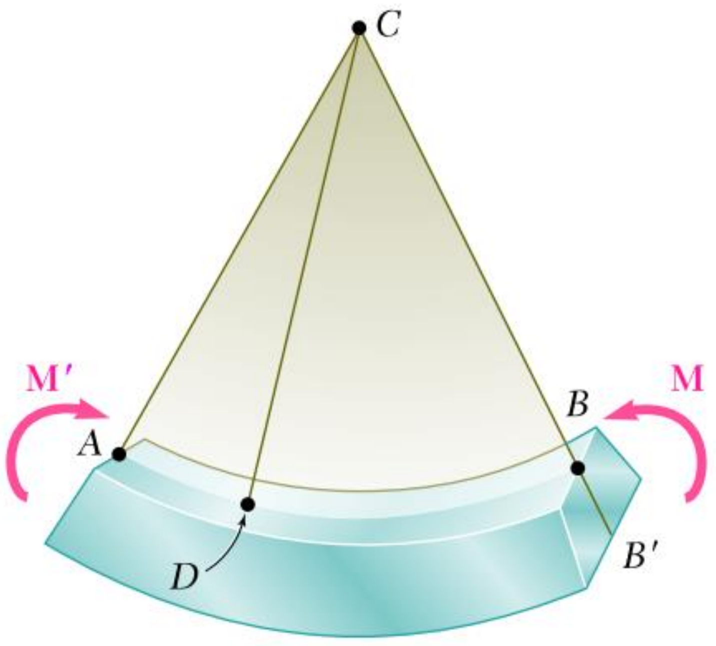
- Once more, note that the actual distribution of stresses in a given cross section cannot be determined from statics alone. It is *statically indeterminate* and may be obtained only by analyzing the *deformations* produced in the member

Deformations

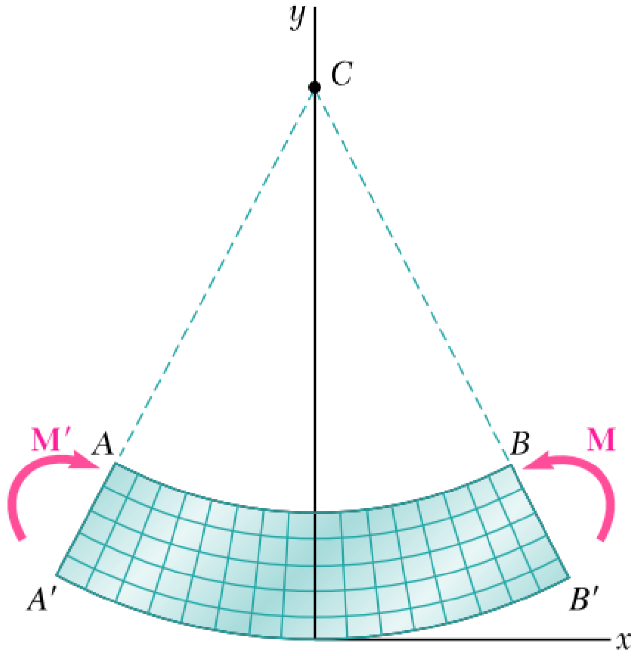
- A prismatic member possessing a plane of symmetry are subjected to equal and opposite couples M and M’ at its ends acting in the plane of symmetry.
- The member will bend under the action of the couples but will remain symmetric with respect to that plane. Moreover, since the bending moment M is the same in any cross section, the member will bend uniformly.

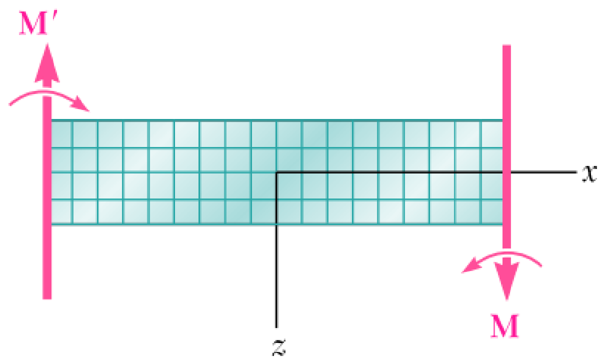


- The line AB along the upper face of the member intersecting the plane of the couples will have a constant curvature. In other words, the line AB will be transformed into a circle of center C, as will the line A'B' along the lower face of the member.



- Note that the line AB will decrease in length when the member is bent (i.e., when $M > 0$), while A'B' will become longer.





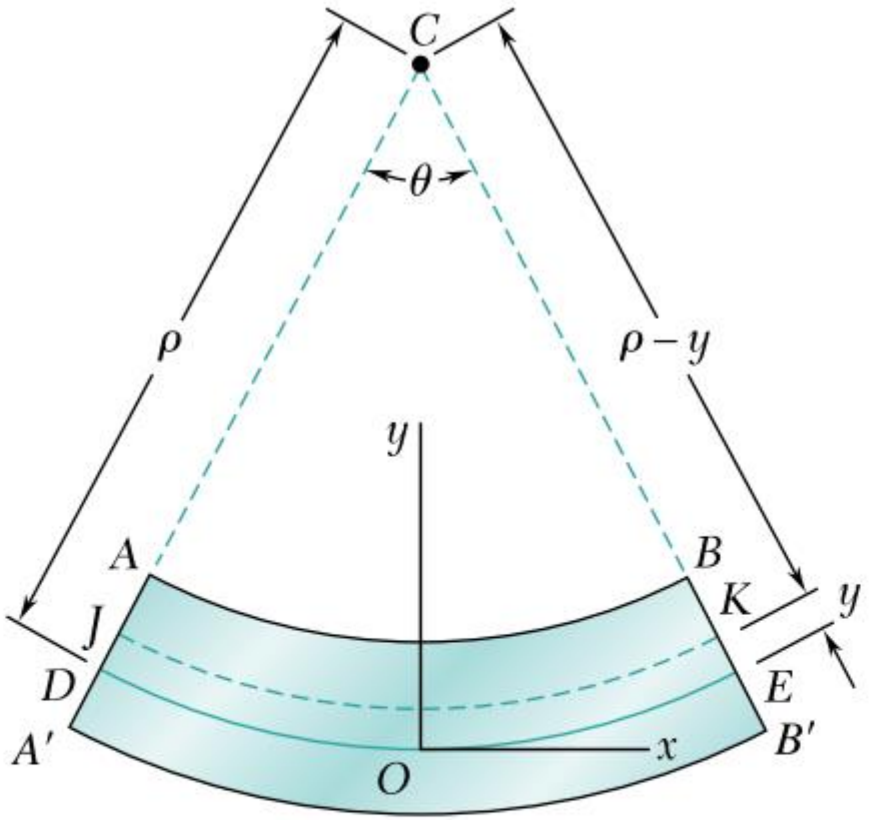
- The cross-section perpendicular to the axis of the member remains plane, and that the plane of the section passes through C.

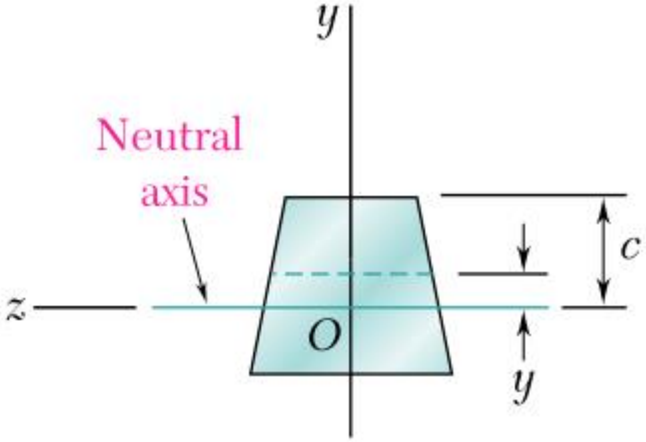
- Suppose that the member is divided into a large number of small cubic elements with faces respectively parallel to the three coordinate planes.

- Since all the faces represented in the two projections of figures are at 90° to each other, we conclude that $\gamma_{xy} = \gamma_{zx} = 0$ and, thus, that $\tau_{xy} = \tau_{zx} = 0$.

- Regarding the three stress components that we have not yet discussed, namely, σ_y, σ_z, and τ_{yz}, we note that they must be zero on the surface of the member.

- We conclude that the only nonzero stress component exerted on any of the small cubic elements considered here is the normal component σ_x. Thus, at any point of a slender member in pure bending, we have a state of uniaxial stress.





- Recalling that, for $M > 0$, lines AB and A′B′ are observed, respectively, to decrease and increase in length, we note that the strain ϵ_x and the stress σ_x are negative in the upper portion of the member (compression) and positive in the lower portion (tension).

- It follows from above that a surface parallel to the upper and lower faces of the member must exist where ϵ_x and σ_x are zero. This surface is called the neutral surface. The neutral surface intersects the plane of symmetry along an arc of circle DE, and it intersects a transverse section along a straight line called the neutral axis of the section.

- Denoting by ρ the radius of arc DE, by θ the central angle corresponding to DE, and observing that the length of DE is equal to the length L of the undeformed member, we write

$$L = \rho\theta$$

- Considering the arc JK located at a distance y above the neutral surface, its length L' is

$$L' = (\rho - y)\theta$$

- Since the original length of arc JK was equal to L, the deformation of JK is

$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

- The longitudinal strain ε_x in the elements of JK is obtained by dividing δ by the original length L of JK. Write

$$\varepsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta} = -\frac{y}{\rho}$$

- The minus sign is due to the fact that it is assumed the bending moment is positive, and thus the beam is concave upward.

- Because of the requirement that transverse sections remain plane, identical deformations occur in all planes parallel to the plane of symmetry. Thus, the value of the strain given by $\varepsilon_x = -\frac{y}{\rho}$ is valid anywhere, and the longitudinal normal strain ε_x varies linearly with the distance y from the neutral surface.

- The strain ε_x reaches its maximum absolute value when y is largest. Denoting the largest distance from the neutral surface as c (corresponding to either the upper or the lower surface of the member) and the maximum absolute value of the strain as ε_m, we have

$$\varepsilon_m = \frac{c}{\rho}$$

- and we can write that

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

- To compute the strain or stress at a given point of the member, we must first locate the neutral surface in the member. To do this, we must specify the stress-strain relation of the material used.

Stresses and Deformations In The Elastic Range

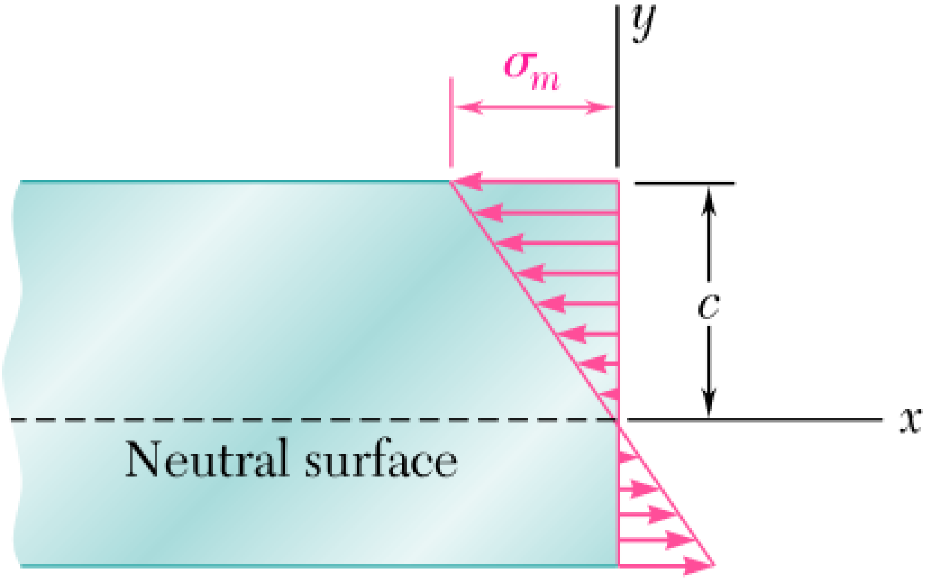
- We now consider the case when the bending moment M is such that the normal stresses in the member remain below the yield strength σ_Y. This means that the stresses in the member remain below the proportional limit and the elastic limit as well. There will be no permanent deformation, and Hooke's law for uniaxial stress applies. Assuming the material to be homogeneous and denoting its modulus of elasticity by E, the normal stress in the longitudinal x direction is

$$\sigma_x = E\varepsilon_x$$

- Recalling $\varepsilon_x = -\frac{y}{c}\varepsilon_m$ and multiplying both members by E, we write

$$E\varepsilon_x = -\frac{y}{c}(E\varepsilon_m) \text{ or } \sigma_x = -\frac{y}{c}\sigma_m$$

- where σ_m denotes the maximum absolute value of the stress. This result shows that, in the elastic range, the normal stress varies linearly with the distance from the neutral surface.



- Note that neither the location of the neutral surface nor the maximum value of the stress have yet to be determined. Both can be found using $\int \sigma_x dA = 0$ and $\int(-y\sigma_x dA) = M$. Substituting for σ_x from $\sigma_x = -\frac{y}{c}\sigma_m$ into $\int \sigma_x dA = 0$, write

$$\int \sigma_x dA = \int \left(\frac{y}{c}\sigma_m\right) dA = -\frac{\sigma_m}{c}\int y dA = 0$$

- from which

$$\int y dA = 0$$

- This equation shows that the first moment of the cross section about its neutral axis must be zero. Thus, for a member subjected to pure bending and as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section.

- Specifying that the z axis coincides with the neutral axis of the cross section, substitute σ_x from $\sigma_x = -\frac{y}{c}\sigma_m$ into $\int(-y\sigma_x dA) = M$:

$$\int(-y)\left(-\frac{y}{c}\sigma_m\right)dA = M$$

$$\frac{\sigma_m}{c}\int y^2 dA = M$$

- Recall that for pure bending the neutral axis passes through the centroid of the cross section and I is the moment of inertia or second moment of area of the cross section with respect to a centroidal axis perpendicular to the plane of the couple M.

$$\sigma_m = \frac{Mc}{I}$$

- Substituting for σ_m from $\sigma_m = \frac{Mc}{I}$ into $\sigma_x = -\frac{y}{c}\sigma_m$, we obtain the normal stress σ_x at any distance y from the neutral axis:

$$\sigma_x = -\frac{My}{I}$$

- Equation is called the elastic flexure formulas, and the normal stress σ_x caused by the bending or "flexing" of the member is often referred to as the flexural stress. The stress is compressive ($\sigma_x < 0$) above the neutral axis ($y > 0$) when the bending moment M is positive and tensile ($\sigma_x > 0$) when M is negative.

- Returning to $\sigma_m = \frac{Mc}{I}$, the ratio I/c depends only on the geometry of the cross section. This ratio is defined as the elastic section modulus S, where

$$Elastic\ section\ modulus = S = \frac{I}{c}$$

- Substituting S for I/c into $\sigma_m = \frac{Mc}{I}$, this equation in alternative form is

$$\sigma_m = \frac{M}{S}$$

- Since the maximum stress σ_m is inversely proportional to the elastic section modulus S, beams should be designed with as large a value of S as is practical.

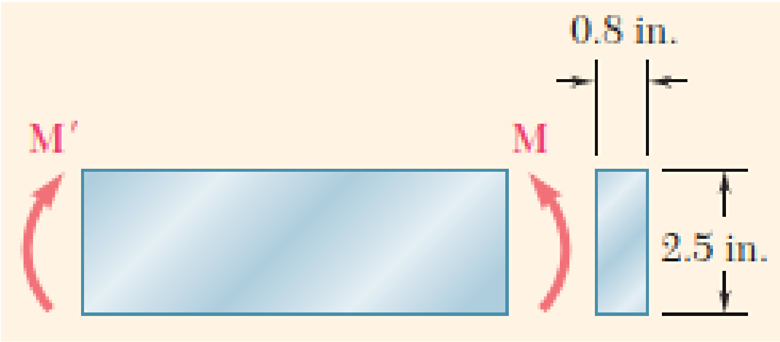
- The deformation of the member caused by the bending moment M is measured by the curvature of the neutral surface. The curvature is defined as the reciprocal of the radius of curvature r and can be obtained by solving $\varepsilon_m = \frac{c}{\rho}$ for $^1/_\rho$:

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c}$$

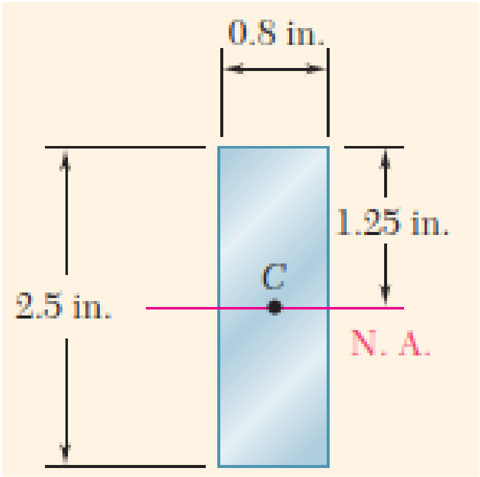
- In the elastic range, $\varepsilon_m = {}^{\sigma_m}/_E$. Substituting ε_m and recalling $\sigma_m = \frac{Mc}{I}$, write

$$\frac{1}{\rho} = \frac{1}{Ec}\frac{Mc}{I} = \frac{M}{EI}$$

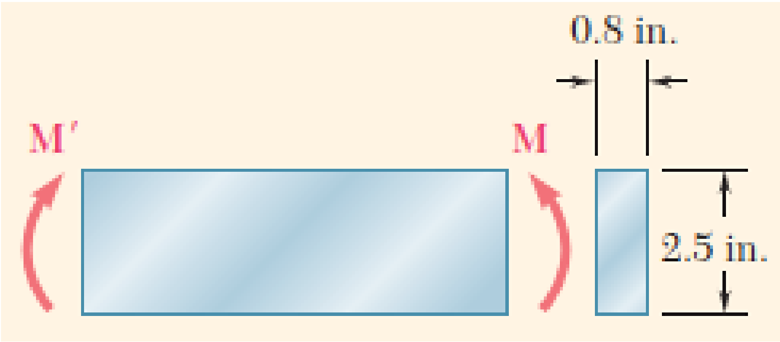
Concept Application 4.1



A steel bar of 0.8 x 2.5-in. rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar. Determine the value of the bending moment M that causes the bar to yield. Assume $\sigma_Y = 36$ ksi.

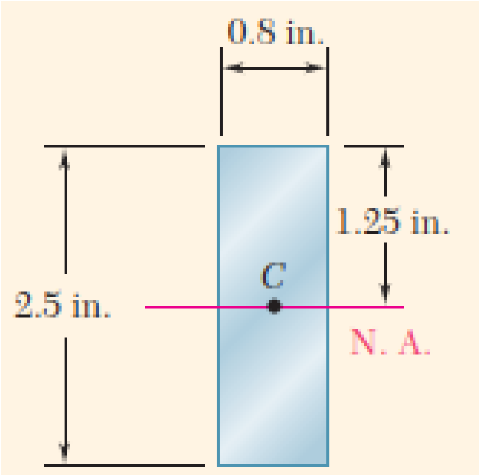


Concept Application 4.1



Since the neutral axis must pass through the centroid C of the cross section, c =1.25 in. On the other hand, the centroidal moment of inertia of the rectangular cross section is

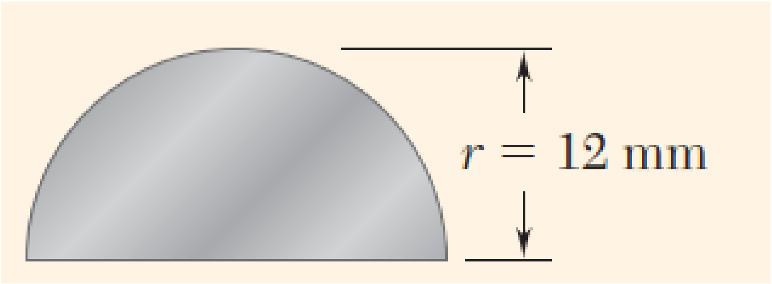
$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.8\text{ in.})(2.5\text{ in.})^3 = 1.042\text{ in}^4$$



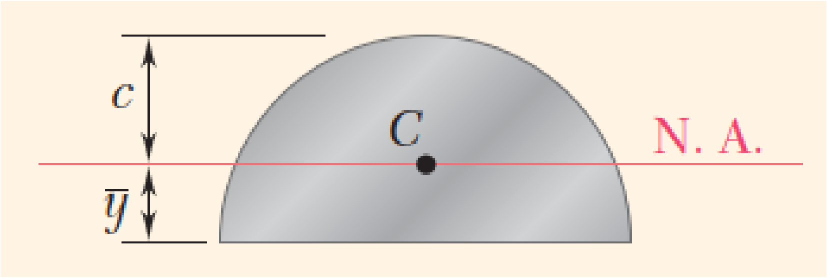
$$\sigma_m = \frac{Mc}{I} \rightarrow M = \sigma_m . \frac{I}{c} = (36\ ksi)\left(\frac{1.042\ in^4}{1.25\ in}\right)$$

$$M = 30\ kip.in$$

Concept Application 4.2



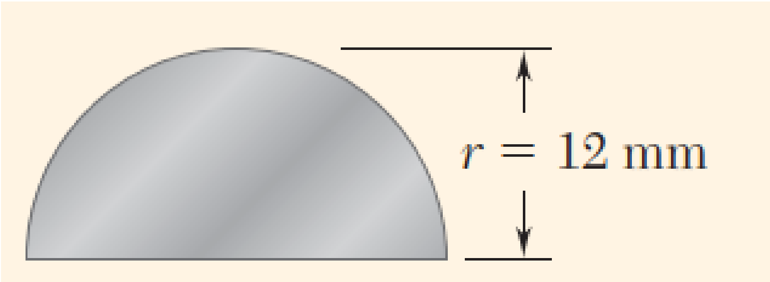
An aluminum rod with a semicircular cross section of radius $r = 12$ mm is bent into the shape of a circular arc of mean radius $r = 2.5$ m. Knowing that the flat face of the rod is turned toward the center of curvature of the arc, determine the maximum tensile and compressive stress in the rod. Use $E = 70$ GPa.



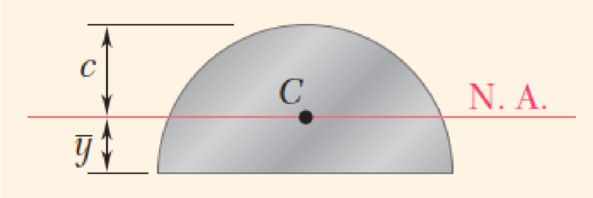
Firstly, the centroid C of the cross-section where the neutral axis passes, must be found

$$\bar{y} = \frac{4r}{3\pi} = \frac{4(12)}{3\pi} = 5.093\ mm$$

Concept Application 4.2



By using Hooke's law and substituting maximum strain ϵ_m value into equation, the maximum normal stress σ_m can be determined. So, the maximum normal strain will be on the farthest point from the centroid (neutral axis). The distance c is then



$$c = r - \bar{y} = 12 - 5.093 = 6.907\ mm$$

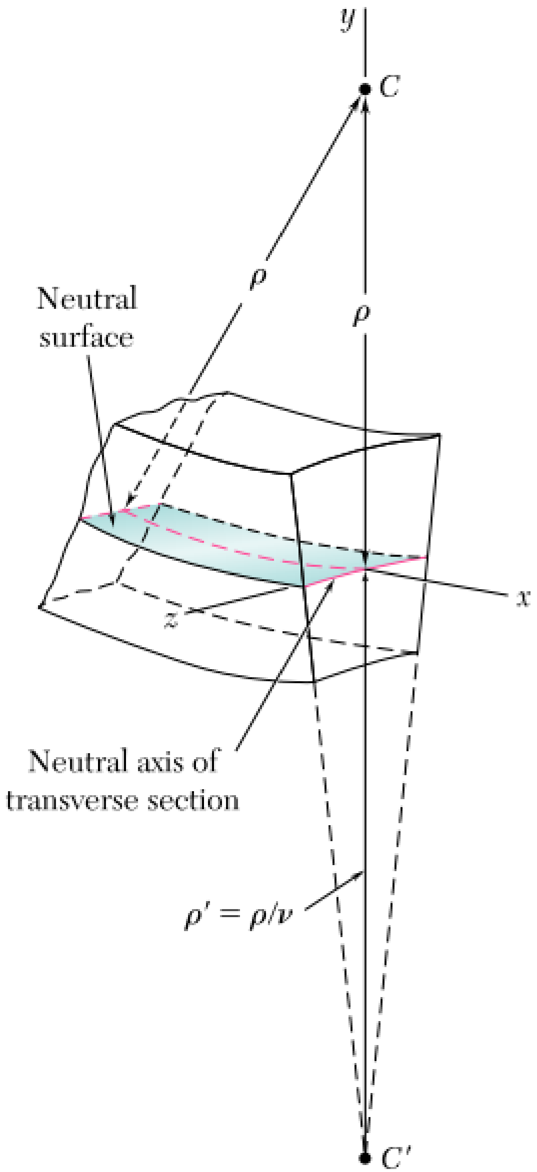
Maximum normal strain, $\epsilon_m = \frac{c}{\rho} = \frac{6.907 * 10^{-3} m}{2.5\ m} = 2.763 * 10^{-3}$

Applying Hooke's law, $\sigma_m = E\epsilon_m = (70 * 10^9\ Pa)(2.763 * 10^{-3}) = 193.4\ MPa$

Maximum compressive stress will be on the flat side of the rod,

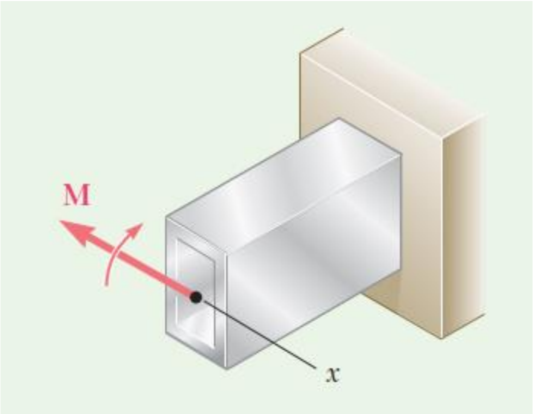
$$\sigma_{comp} = -\frac{\bar{y}}{c}\sigma_m = -\frac{5.093\ mm}{6.097\ mm}(193.4\ MPa) \quad \rightarrow \sigma_{comp} = -142.6\ MPa$$

Deformations in a Transverse Cross Section

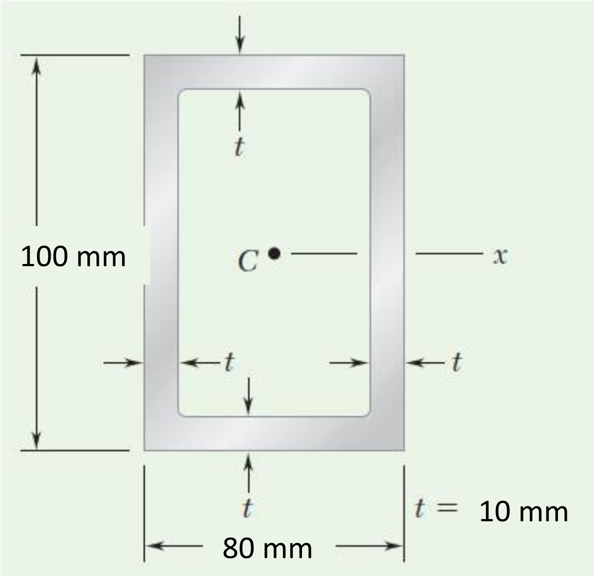


$$Anticlastic\ curvature = \frac{1}{\rho'} = \frac{\nu}{\rho}$$

Sample Problem 4.1



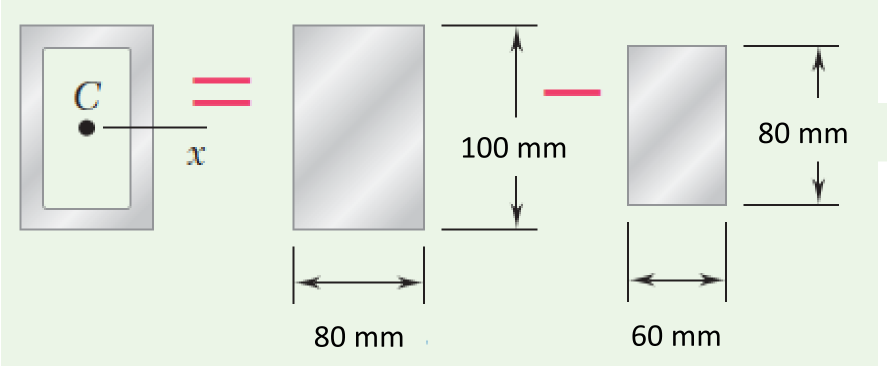
The rectangular tube shown is extruded from an aluminum alloy for which $\sigma_Y = 240\ MPa$, $\sigma_U = 290\ MPa$, and $\mathrm{E} = 70\ GPa$. Neglecting the effect of fillets, determine (a) the bending moment M for which the factor of safety will be 3.00 and (b) the corresponding radius of curvature of the tube.



Allowable stress. For a factor of safety of 3 and ultimate stress of 290 MPa

$$\sigma_{all} = \frac{\sigma_U}{F.S} = \frac{290\ MPa}{3} = 96.67\ MPa$$

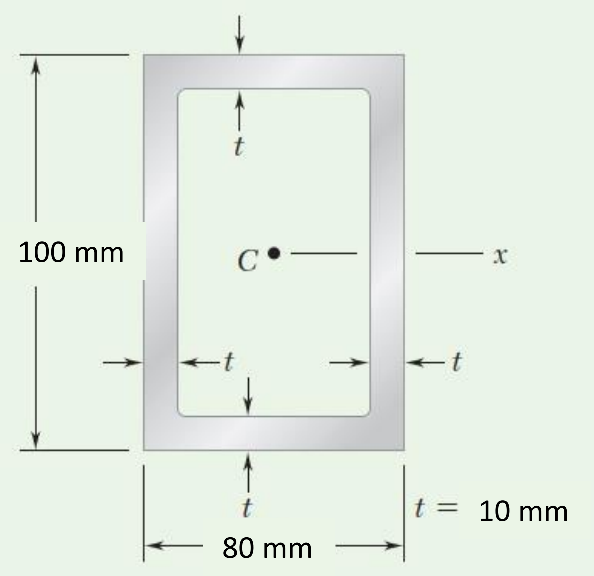
Since $\sigma_{all} < \sigma_Y$ material in elastic range



Moment of Inertia.

$$I = \frac{1}{12}(80)(100)^3 - \frac{1}{12}(60)(80)^3 = 4.1067x10^6\ mm^4$$

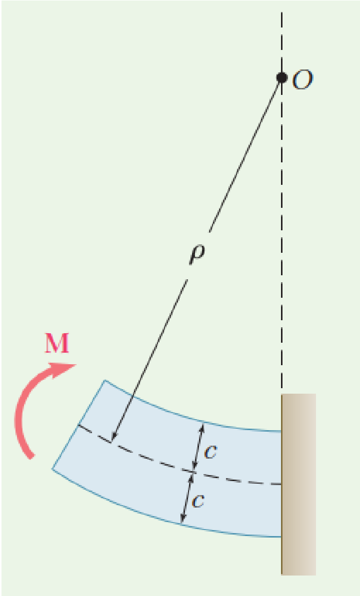
Sample Problem 4.1



Bending moment M. Where $c = \frac{100\ mm}{2} = 50\ mm$

$$\sigma_{all} = \frac{Mc}{I} \rightarrow M = \frac{I}{c}\sigma_{all} = \frac{4.1067x10^{-6}m^4}{0.050\ m}(96.67x10^6 Pa)$$

$$M = 7.94x10^3\ Nm$$

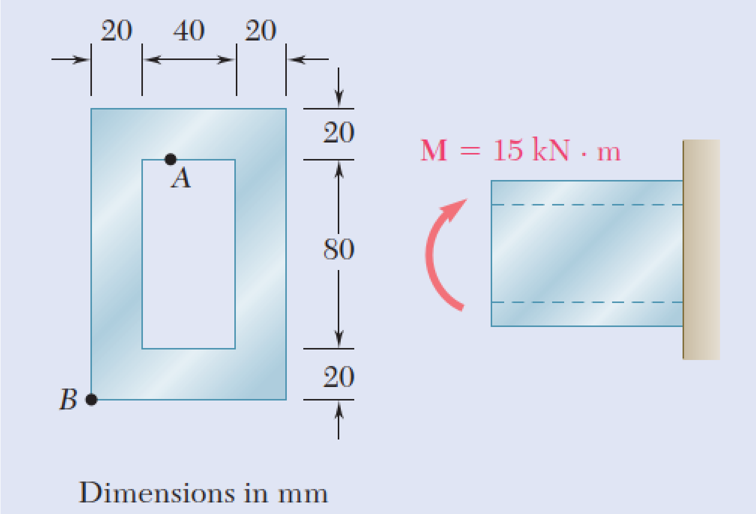


Radius of Curvature.

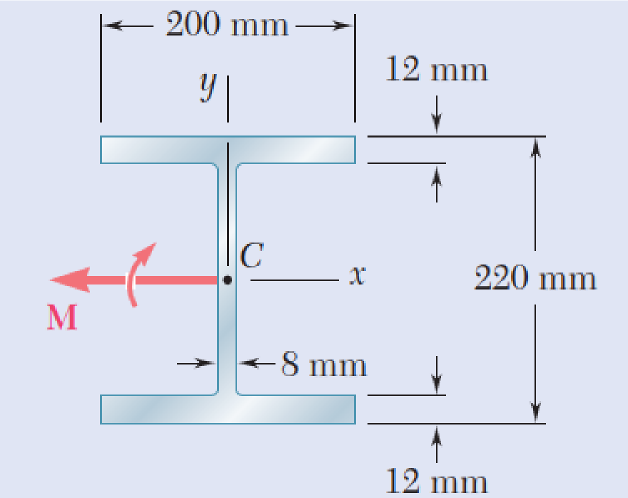
$$\frac{1}{\rho} = \frac{M}{EI} = \frac{7.94x10^3 Nm}{(0.050\ m)(4.1067x10^{-6}m^4)}$$

$$\frac{1}{\rho} = 38.67x10^9\ m^{-1} \rightarrow \rho = 25.86x10^{-12}\ m$$

Problem 4.1. Knowing that the couple shown acts in a vertical plane, determine the stress at (a) point *A*, (b) point *B*.



Problem 4.3. Using an allowable stress of 155 MPa, determine the largest bending moment **M** that can be applied to the wide-flange beam shown. Neglect the effect of fillets.



Problem 4.9. Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.