ME 209
Numerical Methods

2. Numerical Error Analysis & Taylor Series

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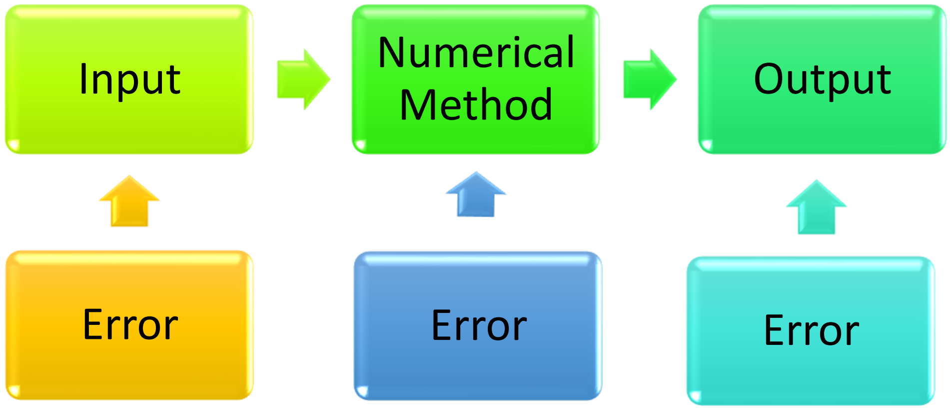
Problem Solving Procedure



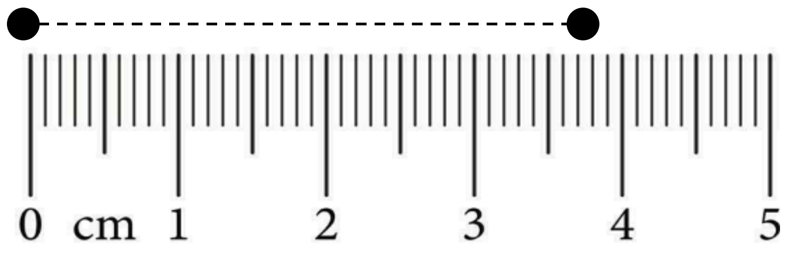
The numerical method used to solve an equation or problem reaches a numerical solution by processing certain input data. Numerical methods, unlike analytical solutions, operate using numbers and contain a certain margin of error. In addition, the input data may be slightly inaccurate. If the input data is found as a result of experiments, these values will contain a margin of error depending on the sensitivity of the measurements made.

Errors

- The fact that the input information and the method used contain errors indicates that the results will contain a certain amount of error. In other words, the results found by numerical methods are *approximate values*, *not exact values*.
- Therefore, error analysis is an important issue in numerical solution. Rather than the results containing errors, it is important that these errors are within acceptable limits, that is, the errors are smaller than the given tolerance value.
- Error is essentially the difference between the actual value and the calculated approximate value.

SIGNIFICANT FIGURES

- The concept of a **significant figure**, or ***digit***, has been developed to formally designate the reliability of a numerical value. The significant digits of a number are those that can be used with confidence. They correspond to **the number of certain digits plus one estimated digit.**
- Consider the problem of measuring the distance between two points using a ruler that has a scale with 1 mm between the finest divisions·



- If we record our measurements in centimeters and if we estimate fractions of a millimeter, then a distance recorded as 3.76 cm gives **two precise** digits (i.e., the 3 and the 7) and **one estimated** digit (i.e., the 6).
- Here, the measurement has **three significant digits**.
- If we recorded the number as 3.762, we would still have only three significant digits since the 2 is not precise.

- The concept of significant figures has two important implications for our study of numerical methods:

1. As introduced in the falling parachutist problem, numerical methods yield approximate results. We must, therefore, develop criteria to specify how confident we are in our approximate result. One way to do this is in terms of significant figures. For example, we might decide that our approximation is acceptable if it is correct to four significant figures.
2. Although quantities such as π, e, or $\sqrt{7}$ represent specific quantities, they cannot be expressed exactly by a limited number of digits. For example,

$$\pi = 3.141592653589793238462643\ldots$$

ad infinitum. Because computers retain only a finite number of significant figures, such numbers can never be represented exactly. The omission of the remaining significant figures is called ***round-off error***.

- When performing computations, the following is a general rule on setting the number of significant digits in a computed value: ***Any mathematical operation using an imprecise digit is imprecise.***
- *Consider the following multiplication of two numbers (4.2**6** and 8.3**9**), each having three significant digits, with the last digit of each being imprecise:*

4.2**6**	Starting number
8.3**9**	Starting number
0.**3834**	0.0**9** times 4.2**6**
1.27**8**	0.3 times 4.2**6**
34.0**8**	8 times 4.2**6**
35.**7414**	Total (product result)

The digits that depend on imprecise digits are underlined· In the final answer, only the first two digits (35) are not based on imprecise digits. Since one and only one imprecise digit can be considered as significant, the result is recorded as 35.7.

Accuracy and Precision

Accuracy: How closely a computed or measured value agrees with the true value.

Precision: How closely computed or measured values agree with each other.

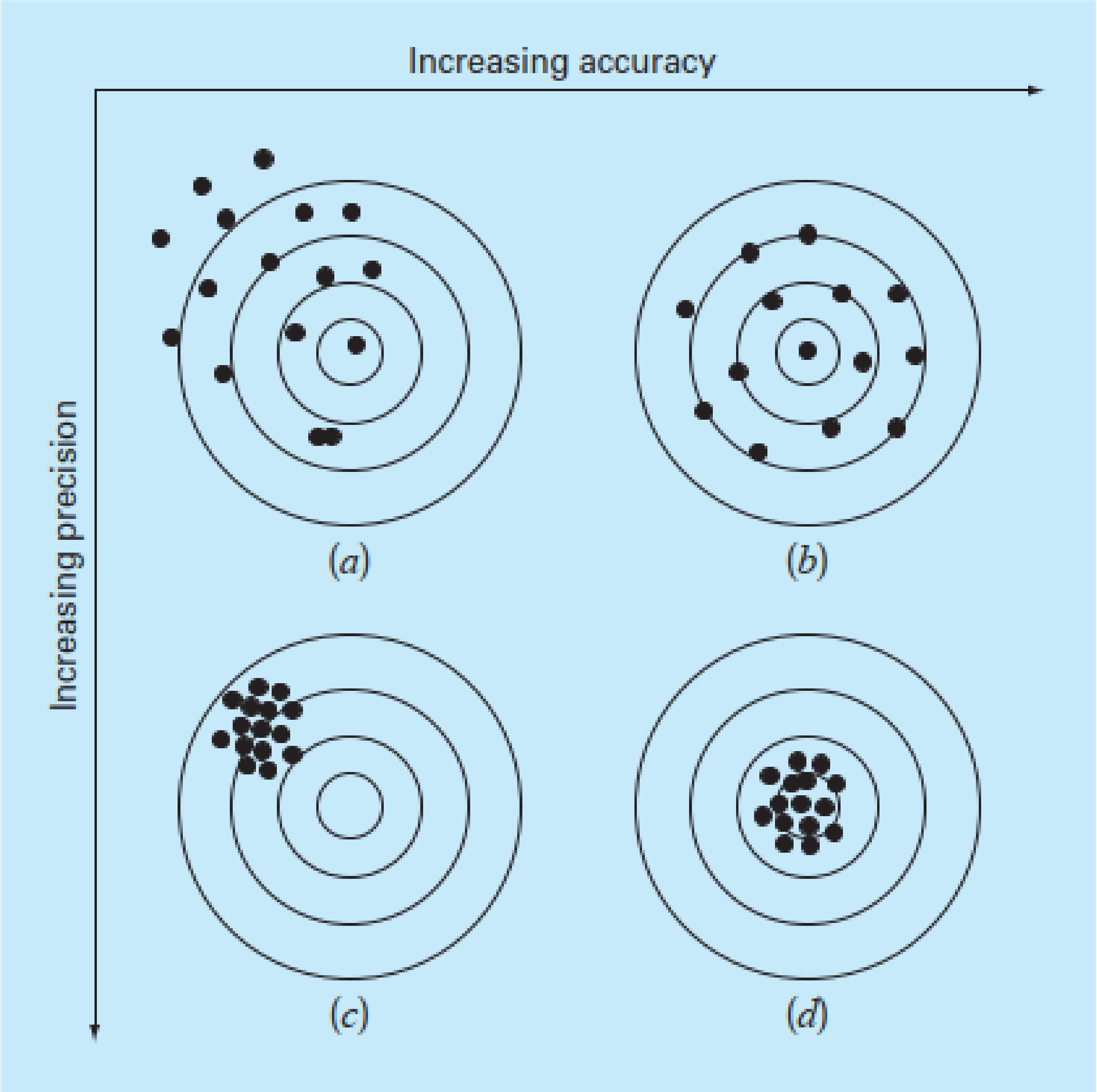


Figure. An example from marksmanship illustrating the concepts of accuracy and precision. (a) Inaccurate and imprecise; (b) accurate and imprecise; (c) inaccurate and precise; (d) accurate and precise.

Error Definitions

Numerical errors arise from the use of approximations to represent exact mathematical operations and quantities. These include ***truncation errors***, which result when approximations are used to represent exact mathematical procedures, and ***round-off errors***, which result when numbers having limited significant figures are used to represent exact numbers.

TRUE ERROR:

True value = approximation + error ⇒ E_t = True value - approximation

Relative True Error (fractional) ⇒ $\varepsilon_t = \frac{E_t}{True\,Value}$

Relative True Error (percentage) ⇒ $\varepsilon_t = \frac{E_t}{True\,Value}100\%$ (Preferred)

Error Definitions

APPROXIMATE ERROR:

Approx. Error: ➡ E_a= Current Approx.- Previous Approx.

Relative Approx. Error (fractional) ➡ $\varepsilon_a = \frac{E_a}{Current\ Approx.}$

Relative Approx. Error (percentage) ➡ $\varepsilon_a = \frac{E_a}{Current\ Approx.}100\%$ (Preferred)

Tolerance: Many numerical methods work in an iterative fashion. There should be a stopping criteria for these methods. We stop when the error level drops below a certain tolerance value (ε_s) that we select (| ε_a | < ε_s)

If this relationship holds, our result is assumed to be within the prespecified acceptable level ε_s .

Error Definitions

EXAMPLE 1. Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (a) the true error and (b) the true percent relative error for each case.

Solution. (a) The error for measuring the bridge is $E_t = 10000 - 9999 = 1\ cm$

and for the rivet is: $E_t = 10 - 9 = 1\ cm$

(b) The percent relative error for the bridge is $\varepsilon_t = \dfrac{E_t}{True\ Value} 100\% = \dfrac{1}{10000} 100\% = 0.01\%$

and for the rivet is: $\varepsilon_t = \dfrac{1}{10} 100\% = 10\%$

Error Definitions

Scarborough criteria: If the tolerance is selected to be $\boldsymbol{\varepsilon_s = 0.5x10^{2-n}\%}$ than the approximation is guaranteed to be correct to *at least n significant figures (digits)*.

EXAMPLE 2. Let's calculate the value of $e^{0.5}$ by using *Maclaurin series expansion*.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

Solution. Scarborough criteria can be employed to determine the error criterion that ensures a result is correct to at least three significant figures:

$$\varepsilon_s = 0.5x10^{2-3}\% = 0.05\%$$

Thus, we will add terms to the series until ε_a falls below this level.

Error Definitions

The first estimate $e^{0.5} = 1$

The second estimate $e^{x} = 1 + x \rightarrow \quad e^{0.5} = 1 + 0.5 = 1.5$

True percent relative error: $\varepsilon_t = \frac{True\ Error}{True\ Value} 100\% = \frac{1.648721 - 1.5}{1.648721} 100\% = 9.02\%$

Approx. percent relative error: $\varepsilon_a = \frac{Approx.\ Error}{Current\ Approx.} 100\% = \frac{1.5 - 1}{1.5} 100\% = 33.3\%$

Since ε_a is not less than the required value of ε_s ,we would continue the computation by adding another term, $x^2/2!$, and repeating the error calculations.

Error Definitions

Terms	Result	ε_t (%)	ε_a (%)
1	1	39.3	
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158

Error Definitions

Round-Off Errors

- Computers can not use infinitely many digits to store numbers.
- Conversion from base 10 to base 2 may create problems.

$$(0.1)_{10} = (0.00011\ 00011\ 00011\ 00011\ \ldots\ldots)_2$$

- Some numbers like π or 1/3 can not be represented exactly.
- Floating point numbers can be stored as single (7-8 digits) or double precision (15-16).

Double precision storage reduces round-off errors.

- Round-off errors can not be totally eliminated but clever algorithms may help to minimize them.
- Round-off errors have accumulative behavior.

Truncation Errors

- Due to the use of an approximation in place of an exact mathematical procedure.
- For example, calculating *sine* of a number using finite number of terms from the infinite

series will result in truncation error.

Error Definitions

Example 3. Calculate $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ for $x = \frac{\pi}{2}$

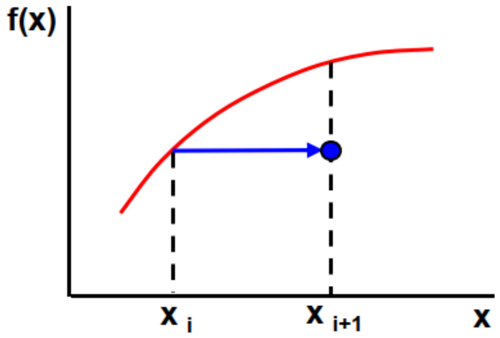
Stop when $\varepsilon_a = 0.001\%$

Number of terms	$\sin(x)$	$\lvert\varepsilon_t\rvert$ %	$\lvert\varepsilon_a\rvert$ %
1	1.570796327	57.1	----
2	0.924832229	7.52	69.84
3	1.004524856	0.45	7.93
4	0.999843101	1.57 E-1	0.47
5	1.000003543	3.54 E-3	0.16 E-1
6	0.999999943	5.63 E-5	3.60 E-4

Important: Round-off and truncation errors generally appear together. As we add more terms, truncation error drops. But at some point, round-off error starts to dominate due to its accumulative behavior and total error will start to increase.

Taylor Series of Expansion (!!)

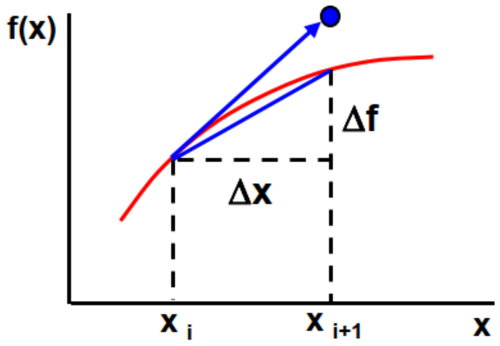
Taylor series is the basics of this course. It is simply used to evaluate a function at one point, using the value of the function and its derivatives at another point.



Known: $f(x_i)$, $f'(x_i)$, $f''(x_i)$, etc.

Unknown: $f(x_{i+1})$

0th order approximation: $f(x_{i+1}) \approx f(x_i)$



Known: $f(x_i)$, $f'(x_i)$, $f''(x_i)$, etc.

Unknown: $f(x_{i+1})$

1st order approximation: $$f'(x_i) = \frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$f(x_{i+1}) \approx f(x_i) + h\, f'(x_i)$$

- $h = x_{i+1} - x_i$ is called the **step size**.
- In general approximations for $f(x_{i+1})$ gets better as the order of approximation increases and as as h decreases.

Taylor Series of Expansion (!!)

Generalization of Taylor Series

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + f''(x_i)\frac{h^2}{2!} + f'''(x_i)\frac{h^3}{3!} + \cdots + f^n(x_i)\frac{h^n}{n!} + R_n$$

- This is the n*th* order Taylor series approximation of $f(x_{i+1})$ around x_i.
- R_n *is the remainder (truncation error).*

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

- *n*[th] *order Taylor series expansion will be exact if f(x) is an nth order polynomial.* R_n *will have*
- *(n+1)*[th] *derivative which is zero.*

Taylor Series of Expansion (!!)

The square-root function, using the Taylor series expansion, can be expressed as

$$f(x) = \sqrt{x}$$

To evaluate the Taylor series, the derivatives of the function are developed

$$f^{(1)}(x) = \frac{1}{2} x^{-0.5} \qquad f^{(2)}(x) = -\frac{1}{4} x^{-1.5} \qquad f^{(3)}(x) = \frac{3}{8} x^{-2.5}$$

For a base point x_0 = 1 and h = 0.001, the four terms of the Taylor series produce the following estimate for the square root of 1.001:

$$\mathrm{f}(1.001) = \mathrm{f}(1) + \mathrm{f}'(1).\,0.001 + \mathrm{f}''(1)\,\frac{0.001^2}{2!} + \mathrm{f}'''(1)\,\frac{0.001^3}{3!}$$

$$f(1.001) = \sqrt{1.001} \approx \sqrt{1} + 0.5(0.001)(1)^{-0.5} - \frac{1}{4(2!)}(0.001)^2(1)^{-1.5} + \frac{3}{8(3!)}(0.001)^3(1)^{-2.5}$$

$$f(1.001) \approx 1 + 0.5 \times 10^{-3} - 0.125 \times 10^{-6} + 0.625 \times 10^{-10} = 1.0004999$$

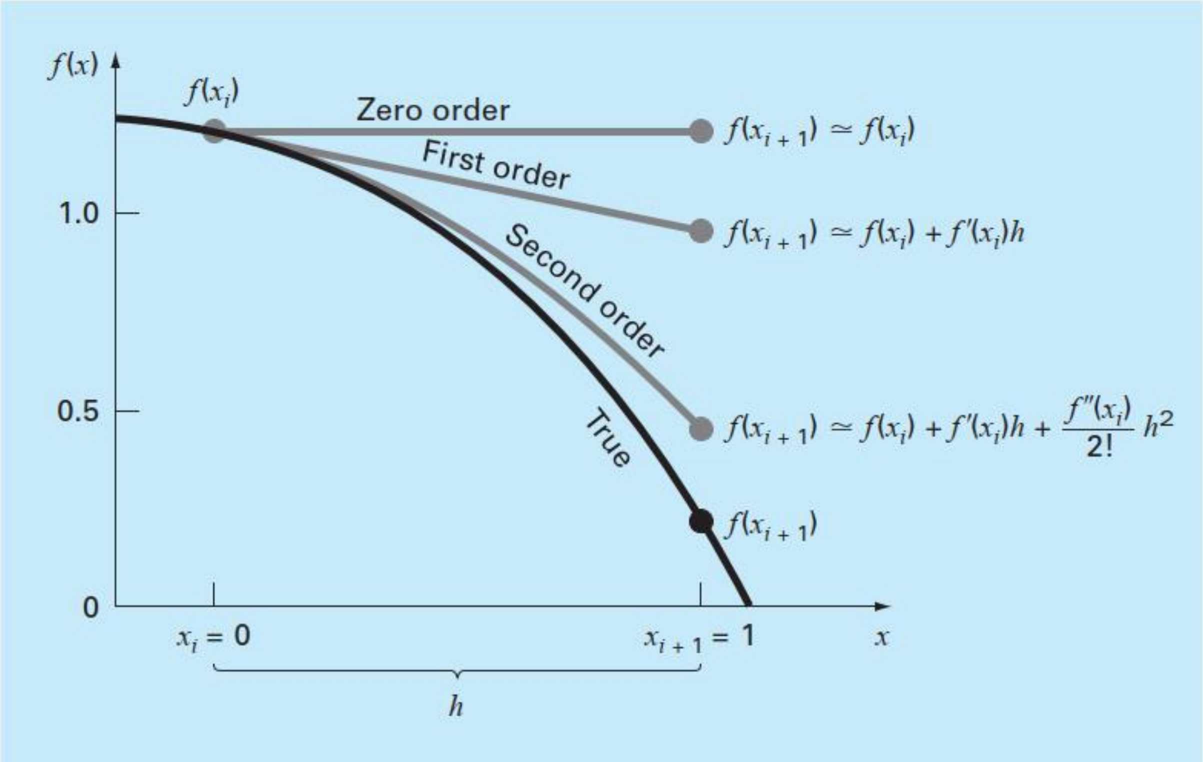
Taylor Series (!!)

Example: Use zero- through fourth-order Taylor series expansions to approximate the function

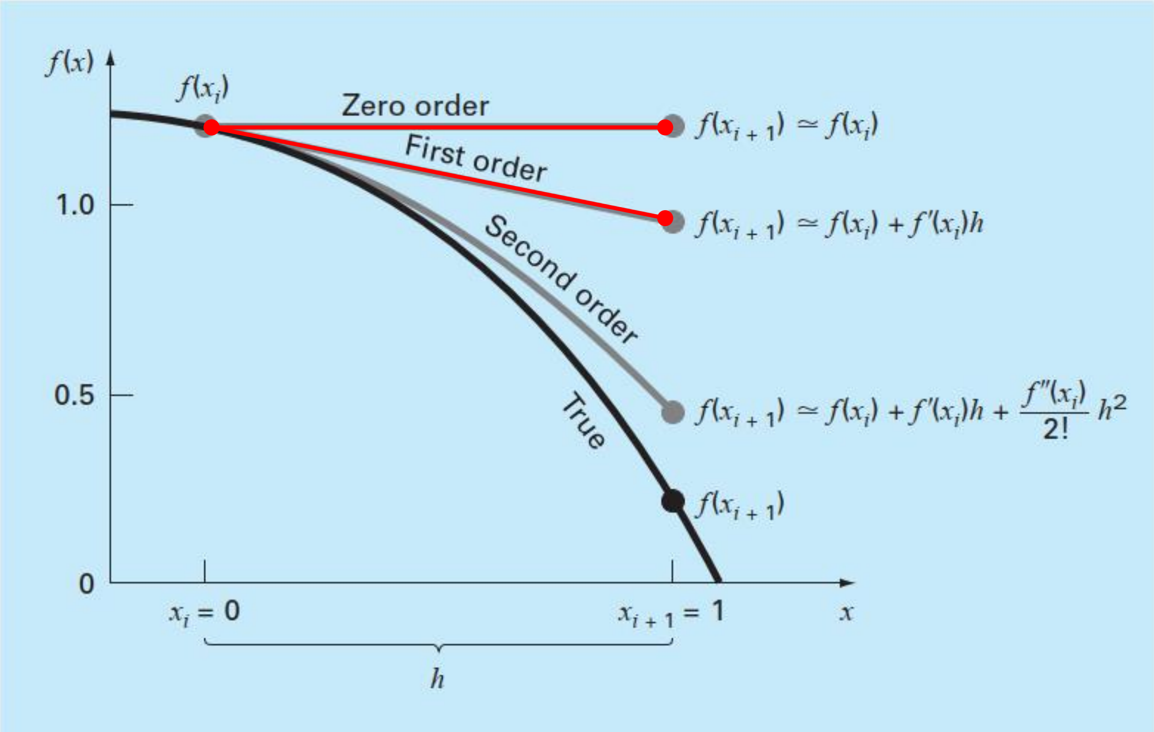
$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

from $x_i = 0$ with *h=1*. That is, predict the function's value at $x_{i+1} = 1$.

- Because we are dealing with a known function, we can compute values for f(x) between 0 and 1. Thus, the true value that we are trying to predict is 0.2.



Taylor Series (!!)



$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

- Taylor series approximation with *n=0,* zero-order appr.

$$f(x_{i+1}) \cong 1.2$$

Truncation error: $E_t = 0.2 - 1.2 = -1.0$

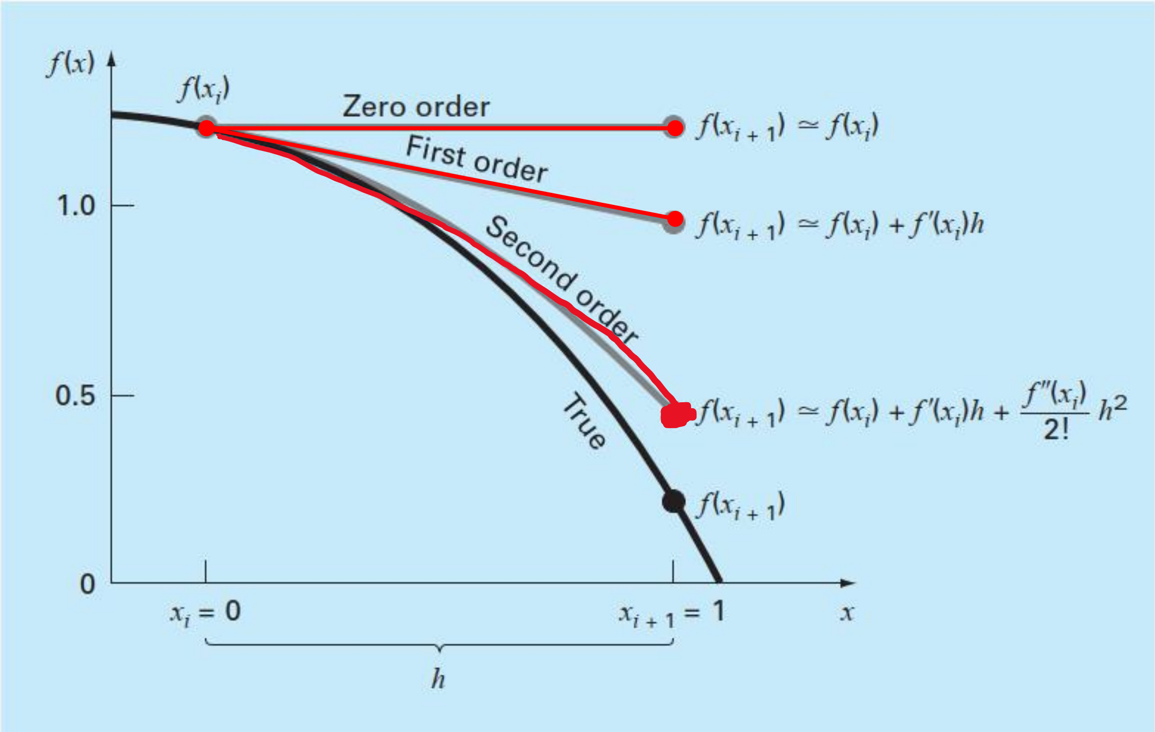
- For *n=1,* the first derivative must be determined and evaluated at *x = 0:*

$$f'(0) = -0.4(0)^3 - 0.45(0)^2 - 1(0) - 0.25 = -0.25$$

Thus, the first-order appr. $f(x_{i+1}) \cong 1.2 - 0.25h \rightarrow f(1) \cong 0.95$

Truncation error: $E_t = 0.2 - 0.95 = -0.75$

Taylor Series (!!)



$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

- For *n=2, the second-order derivative:*

$$f''(0) = -1.2(0)^2 - 0.9(0) - 1 = -1$$

Thus, the second-order appr.

$$\mathrm{f}(x_{i+1}) \cong 1.2 - 0.25h - 0.5h^2 \rightarrow \mathrm{f}(1) \cong 0.45$$

Truncation error: $E_t = 0.2 - 0.45 = -0.25$

- Additional terms would improve the approximation even more. In fact, the inclusion of the third and the fourth derivatives results in the same value.

Taylor Series (!!)

- In general, we can usually assume that the ***truncation error is decreased by the addition of terms to the Taylor series***. In many cases, if h is sufficiently small, the first- and other lower-order terms usually account for a disproportionately high percent of the error. Thus, only a few terms are required to obtain an adequate estimate.

Example: Use Taylor series expansions with $n = 0$ to 6 to approximate $f(x) = \cos x$ at $x_{i+1} = \pi/3$ on the basis of the value of f(x) and its derivatives at $x_i = \pi/4$.

Be careful, this means that $\boldsymbol{h = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}}$

Zero-order appr. $\quad \mathrm{f}(\pi/3) \cong \cos\left(\frac{\pi}{4}\right) = 0.707106781$

Percent relative error: $\quad \varepsilon_t = \frac{0.5 - 0.707106781}{0.5} = -41.4\%$

First-order appr. $\quad \mathrm{f}(\pi/3) \cong \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) = 0.521986659 \qquad \varepsilon_t = -4.40\%$

Taylor Series (!!)

Second-order appr. $f(\pi/3) \cong \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) - \cos\left(\frac{\pi}{4}\right).\frac{1}{2!}.\left(\frac{\pi}{12}\right)^2 = 0.497754491$

$\varepsilon_t = 0.449\%$

Order n	$f^{(n)}(x)$	$f(\pi/3)$	ε_t
0	cos x	0.707106781	−41.4
1	−sin x	0.521986659	−4.4
2	−cos x	0.497754491	0.449
3	sin x	0.499869147	2.62×10^{-2}
4	cos x	0.500007551	-1.51×10^{-3}
5	−sin x	0.500000304	-6.08×10^{-5}
6	−cos x	0.499999988	2.44×10^{-6}

- Notice that the derivatives never go to zero as was the case with the polynomial. Therefore, each additional term results in some improvement in the estimate.
- Also notice how most of the improvement comes with the initial terms.

NEXT WEEK
ROOTS OF EQUATIONS