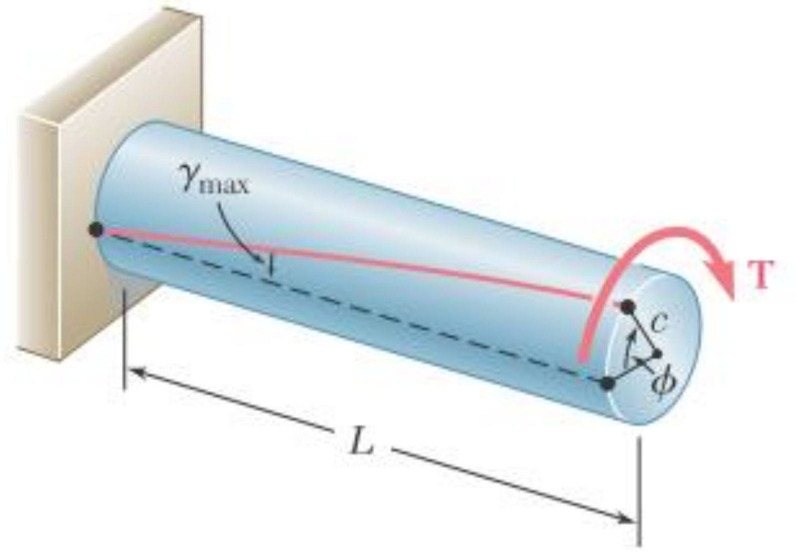
CH.3 TORSION

Part II

Angle Of Twist in the Elastic Range

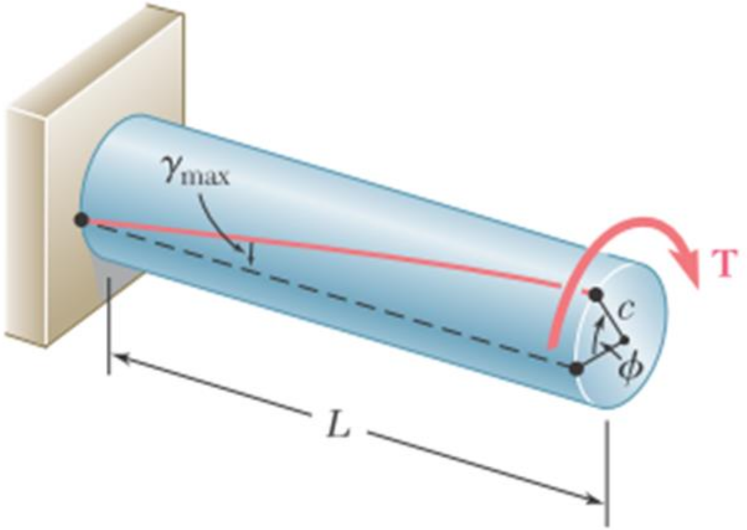
- In this section, a relationship will be determined between the angle of twist ϕ of a circular shaft and the torque T exerted on the shaft. The entire shaft is assumed to remain elastic. Considering first the case of a shaft of length L with a uniform cross section of radius c subjected to a torque T at its free end, recall that the angle of twist ϕ and the maximum shearing strain γ_{max} are related as

$$\gamma_{max} = \frac{c\phi}{L}$$



- But in the elastic range, the yield stress is not exceeded anywhere in the shaft. Hooke's law applies, and $\gamma_{max} = {}^{\tau_{max}}/_{G}$.

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

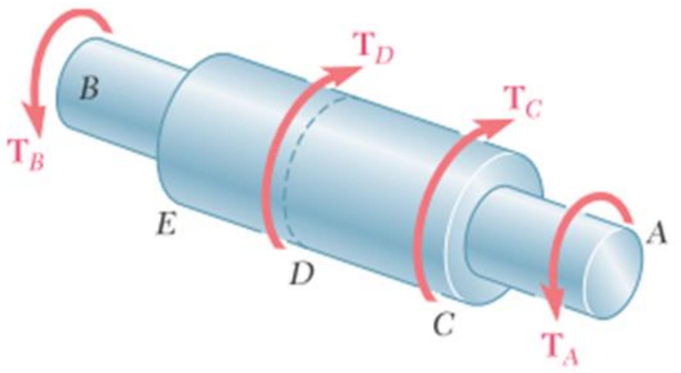


- Equating the right-hand members of $\gamma_{max} = \frac{c\phi}{L}$ and $\gamma_{max} = \frac{Tc}{JG}$ and solving for ϕ, write

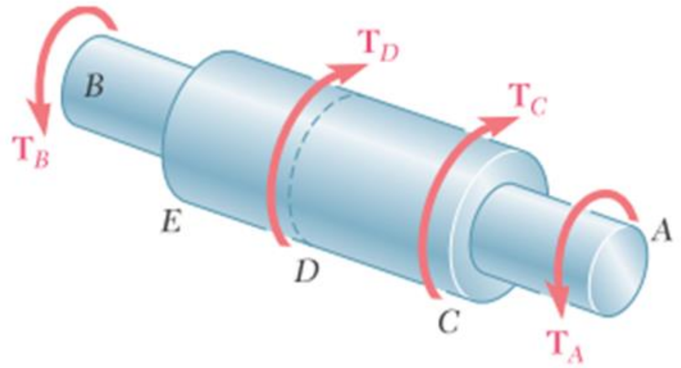
$$\phi = \frac{TL}{JG}$$

- where ϕ is in radians. The relationship obtained shows that, within the elastic range, the angle of twist ϕ is proportional to the torque T applied to the shaft.

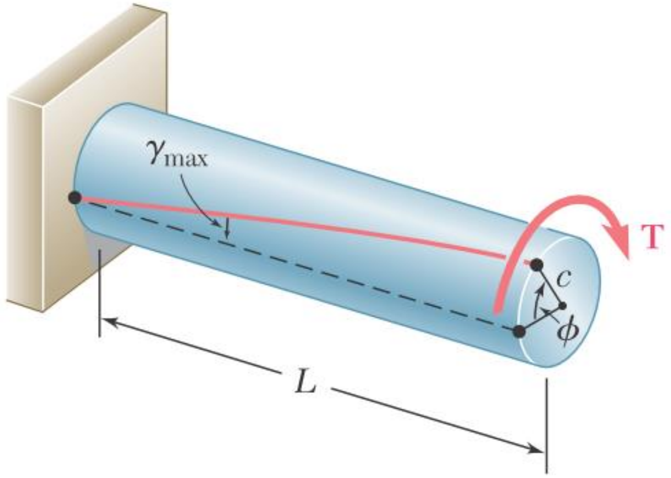
- $\phi = \frac{TL}{JG}$ can be used for the angle of twist only if the shaft is homogeneous (constant G), has a uniform cross section, and is loaded only at its ends. If the shaft is subjected to torques at locations other than its ends or if it has several portions with various cross sections and possibly of different materials, it must be divided into parts that satisfy the required conditions.



- For shaft AB shown in figure, four different parts should be considered: AC, CD, DE, and EB. The total angle of twist of the shaft (i.e., the angle through which end A rotates with respect to end B) is obtained by algebraically adding the angles of twist of each component part. Using the internal torque T_i, length L_i, cross-sectional polar moment of inertia J_i, and modulus of rigidity G_i, corresponding to part i, the total angle of twist of the shaft is



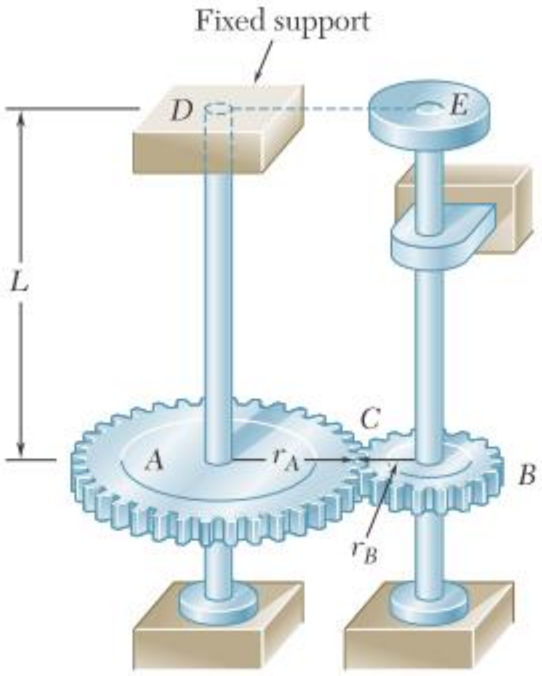
$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

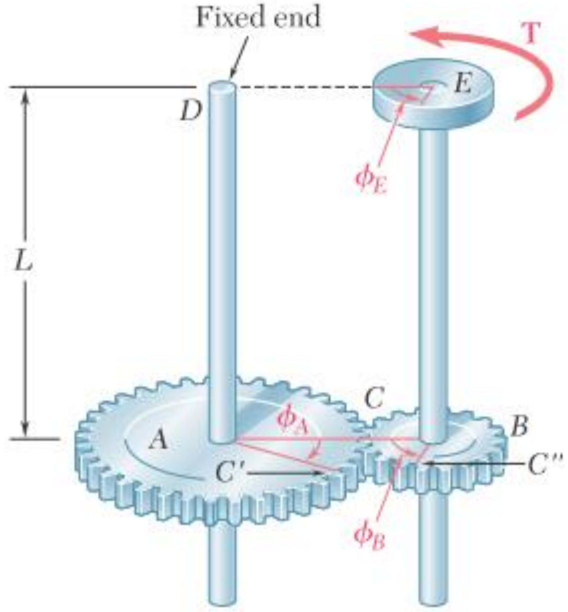


- The shaft shown in figure had one end attached to a fixed support. The angle of twist ϕ was equal to the angle of rotation of its free end.
- When both ends of a shaft rotate, however, the angle of twist of the shaft is equal to the angle through which one end of the shaft rotates *with respect to the other*.

- If a torque T is applied at E, both shafts will be twisted.
  - Since the end D of shaft AD is fixed, the angle of twist of AD is measured by the angle of rotation ϕ_A of end A.
  - On the other hand, since both ends of shaft BE rotate, the angle of twist of BE is equal to the difference between the angles of rotation ϕ_B and ϕ_E (i.e., the angle of twist is equal to the angle through which end E rotates with respect to end B).
  - This relative angle of rotation, $\phi_{E/B}$, is

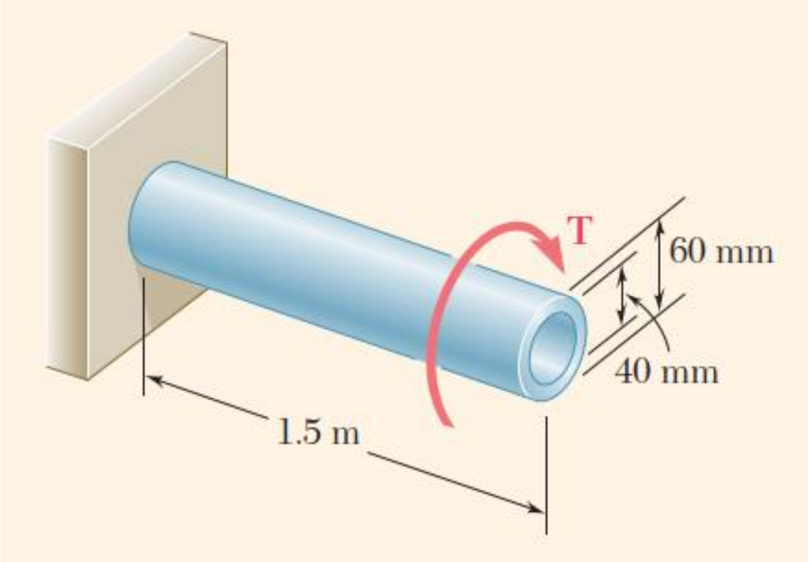
$$\phi_{E/B} = \phi_E - \phi_B = \frac{TL}{JG}$$





Concept Application 3.2

What torque should be applied to the end of the shaft to produce a twist of 2°? Use the value G = 77 GPa for the modulus of rigidity of steel.



$$\phi = \frac{TL}{JG} \quad \rightarrow \quad T = \frac{JG}{L}\phi$$

Substituting the given values

$$\phi = 2^\circ\left(\frac{\pi\ rad}{180^\circ}\right) = 34.9 * 10^{-3}\ rad, \qquad G = 77 * 10^9\ Pa, \qquad L = 1.5\ m,$$

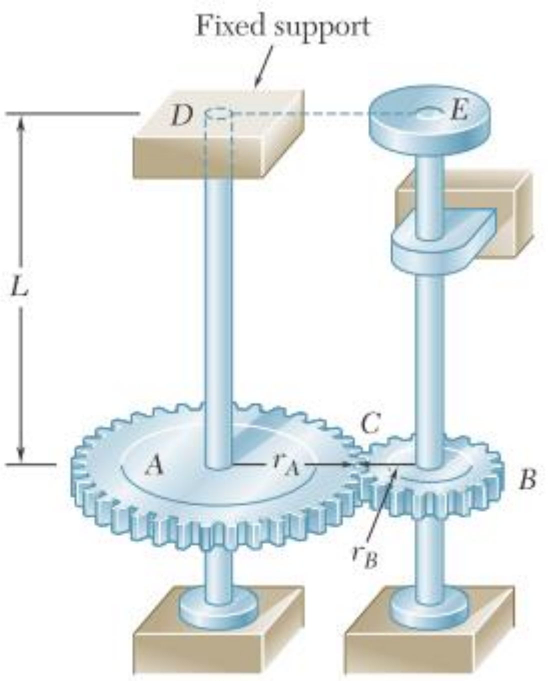
$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = 1.021 * 10^{-6}\ m^4$$

$$T = \frac{JG}{L}\phi = \frac{(1.021 * 10^{-6})(77 * 10^9)}{1.5}(34.9 * 10^{-3})$$

$$\mathrm{T} = 1.829x10^3 N.m$$

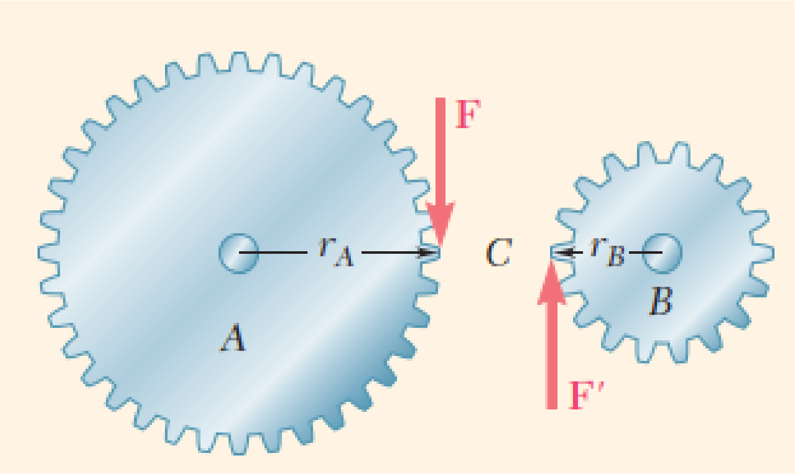
Concept Application 3.4

For the assembly given in the figure, knowing that $r_A = 2r_B$, determine the angle of rotation of end E of shaft BE when the torque **T** is applied at E.



$$\phi_A = \frac{T_{AD}L}{JG}$$

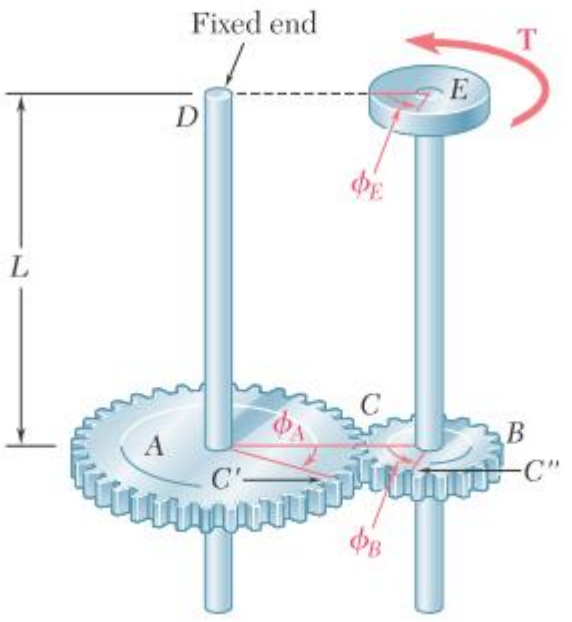
First, the torque T_{AD} must be determined by using the two equal and opposite forces on gears at C.



$r_A = 2r_B$, and $\quad F = F' \rightarrow \frac{T_{AD}}{r_A} = \frac{T}{r_B} \rightarrow T_{AD} = 2T$

Concept Application 3.4

For the assembly given in the figure, knowing that $r_A = 2r_B$, determine the angle of rotation of end E of shaft BE when the torque **T** is applied at E.



$$\phi_A = \frac{T_{AD}L}{JG} = \frac{2TL}{JG} \quad \text{since the arc lengths CC}' = \text{CC}''$$

$$r_A\phi_A = r_B\phi_B \quad \rightarrow \phi_B = \left(\frac{r_A}{r_B}\right)\phi_A = 2\phi_A \quad \rightarrow \phi_B = 2\phi_A = \frac{4TL}{JG}$$

Let's consider shaft BE which both ends are free to rotate, so the angle of twist of the shaft is $\phi_{E/B}$ *through which end E rotates with respect to end B*.

$$\phi_{E/B} = \frac{T_{BE}L}{JG} = \frac{TL}{JG}$$

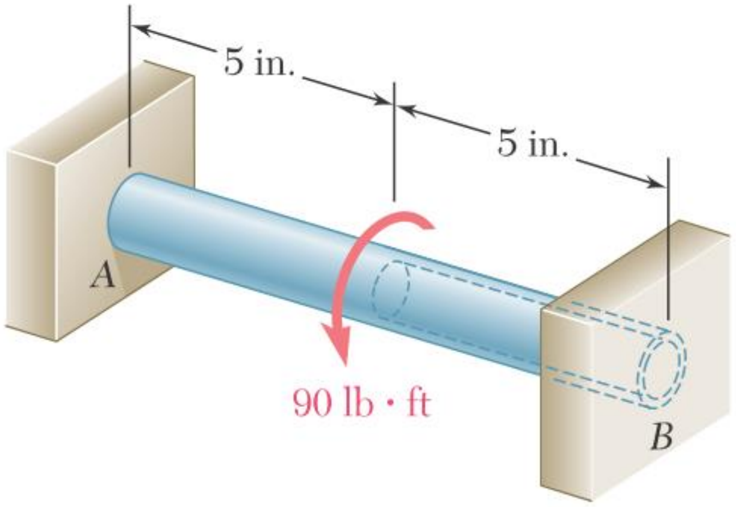
$$\phi_E = \phi_B + \phi_{E/B} = \frac{5TL}{JG}$$

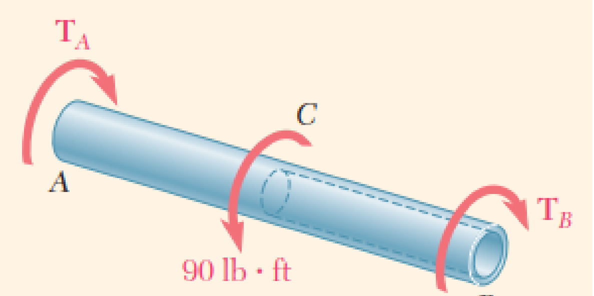
Statically Indeterminate Shafts

- There are situations where the internal torques cannot be determined from statics alone. In such cases, the external torques (i.e., those exerted on the shaft by the supports and connections) cannot be determined from the free-body diagram of the entire shaft. The equilibrium equations must be complemented by relations involving the deformations of the shaft and obtained by the geometry of the problem. Because statics is not sufficient to determine external and internal torques, the shafts are statically indeterminate.

Concept Application 3.5

- A circular shaft AB consists of a 10-in.-long, 7/8-in.-diameter steel cylinder, in which a 5-in.-long, 5/8-in.-diameter cavity has been drilled from end B. The shaft is attached to fixed supports at both ends, and a 90 lb·ft torque is applied at its midsection. Determine the torque exerted on the shaft by each of the supports.





(b)

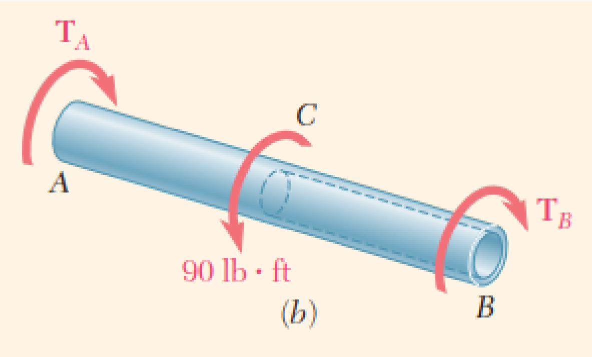
$$T_A + T_B = 90 \text{ lb·ft}$$

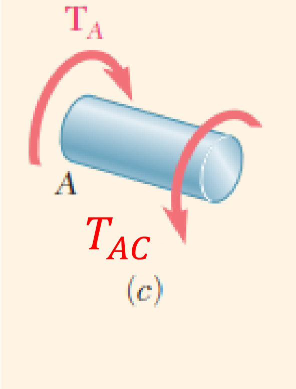
Since this equation is not sufficient to determine the two unknown torques $\mathbf{T}_A$ and $\mathbf{T}_B$, the shaft is statically indeterminate.

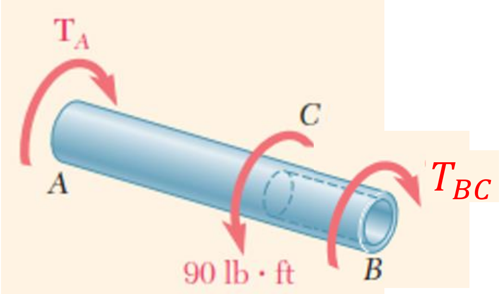
However, $\mathbf{T}_A$ and $\mathbf{T}_B$ can be determined if we observe that the total angle of twist of shaft AB must be zero, since both of its ends are restrained.

$$\phi = \phi_{AC} + \phi_{BC} = 0$$

Concept Application 3.5







$T_A + T_B = 90$ lb·ft

$T_{AC} - T_A = 0 \;\rightarrow T_{AC} = T_A$

$90 - T_A - T_{BC} = 0 \;\rightarrow T_{BC} = 90 - T_A$

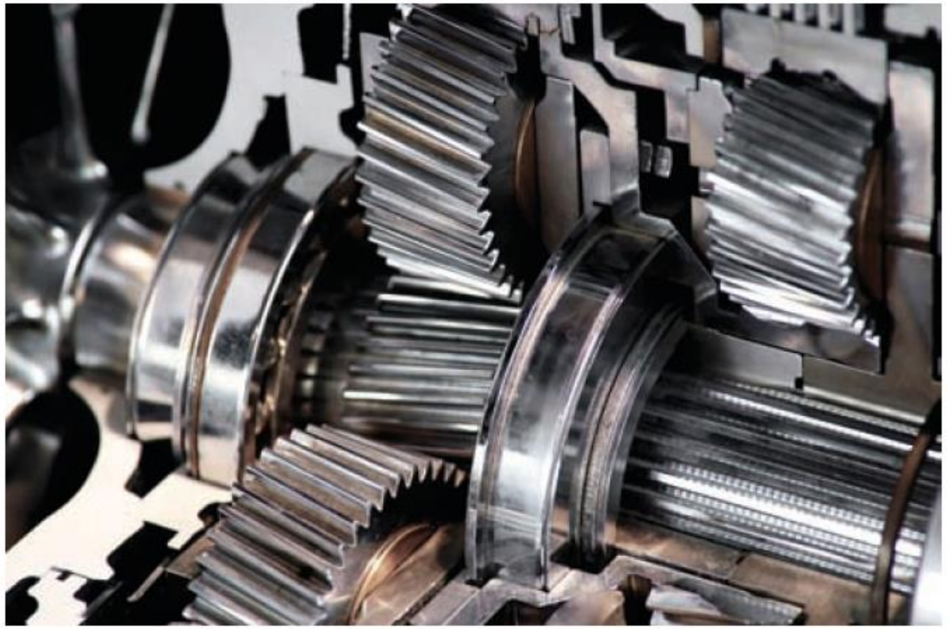
$$\phi = \phi_{AC} + \phi_{BC} = \frac{T_{AC}L}{J_{AC}G} - \frac{T_{BC}L}{J_{BC}G} = \frac{T_A.5}{57.6x10^{-3} * 77x10^9} - \frac{(90 - T_A).5}{42.6x10^{-3} * 77x10^9} = 0$$

$\rightarrow 40.835T_A = 2112.676$

$\rightarrow T_A = 51.74\; lb.ft$

$\rightarrow T_B \; = 38.26\; lb.ft$

Design of Transmission Shafts



- The principal specifications to be met in the design of a transmission shaft are the *power* to be transmitted and the *speed of rotation* of the shaft.
- The role of the designer is to select the material and the dimensions of the cross section of the shaft so that the maximum shearing stress allowable will not be exceeded when the shaft is transmitting the required power at the specified speed.

Power is defined as the work performed per unit of time. Also, the work transmitted by a rotating shaft equals the torque applied times the angle of rotation.

$$P = T\omega$$

In the SI system, power is expressed in *watts* when torque is measured in newton-meters (N.m) and ω is in radians per second (rad/s) (1 W = 1 N.m/s). However, *horsepower* (hp) is often used in engineering practice, where

1 hp = 746 W

$\omega = 2\pi f$, where f is the frequency of the rotation, (i.e., the number of revolutions per second). The unit of frequency is 1 s^{-1} and is called a *hertz* (Hz).

$$\boxed{P = 2\pi fT}$$

- Solving $P = 2\pi fT$ for T, the torque exerted on a shaft transmitting the power P at a frequency of rotation f is

$$T = \frac{P}{2\pi f}$$

- After determining the torque T to be applied to the shaft and selecting the material to be used, the designer carries the values of T and the maximum allowable stress into

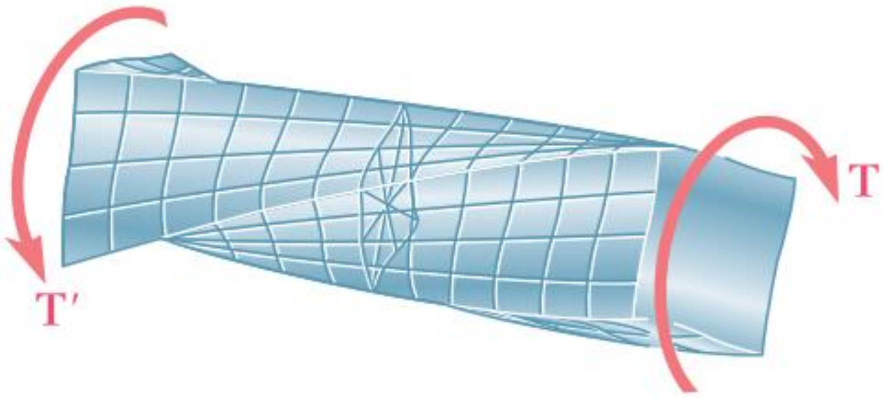
$$\tau_{max} = \frac{Tc}{J}$$

- and finds the the minimum allowable value for the radius of the shaft.

Torsion of Noncircular Members

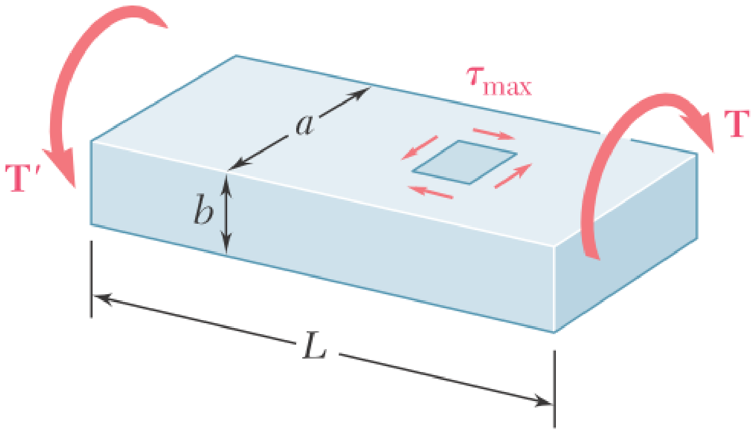
- The formulas obtained for the distributions of strain and stress under a torsional loading up to this time apply only to members with a circular cross section.
- They were derived based on the assumption that the *cross section of the member remained plane and undistorted*. This assumption depends upon the axisymmetry of the member (i.e., the fact that its appearance remains the same when viewed from a fixed position and rotated about its axis through an arbitrary angle).

- A square bar, on the other hand, retains the same appearance only when it is rotated through 90° or 180°. Following a line of reasoning similar to that used in our previous discussions, one could show that the diagonals of the square cross section of the bar and the lines joining the midpoints of the sides of that section remain straight. However, because of the lack of axisymmetry of the bar, any other line drawn in its cross section will deform when it is twisted, and the cross section will be warped out of its original plane.



- Equations used to define the distributions of strain and stress in an elastic circular shaft cannot be used for noncircular members. For example, it would be wrong to assume that the shearing stress in the cross section of a square bar varies linearly with the distance from the axis of the bar and is therefore largest at the corners of the cross section. The shearing stress is actually zero at these points.

- The determination of the stresses in noncircular members subjected to a torsional loading is beyond the scope of this course. However, results obtained from the mathematical theory of elasticity for straight bars with a uniform rectangular cross section are given here for our use.



a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

- The maximum shearing stress occurs along the center line of the wider face and is equal to

$$\tau_{max} = \frac{T}{c_1 a b^2}$$

- The angle of twist can be expressed as

$$\phi = \frac{TL}{c_2 a b^3 G}$$

- Coefficients c_1 and c_2 depend only upon the ratio a/b and are given in Table for a number of values of that ratio.