ME 209
Numerical Methods

10. Numerical Integration

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10.1. MOTIVATION

- In calculus, the *integration* is the inverse process to differentiation.
- According to the dictionary definition, to integrate means “to bring together, as parts, into a whole; to unite; to indicate the total amount . . . “
- Mathematically, the integration is represented by

$$I = \int_a^b f(x)\,dx$$

which stands for the integral of the function f(x) with respect to the independent variable x, evaluated between the limits x = a to x = b.

- The function f(x) in equation above is referred to as the *integrand*.

- As suggested by the dictionary definition, the “meaning” of $I = \int_a^b f(x)\,dx$ is the *total value*, or *summation*, of *f(x) dx* over the range *x* = *a* to *b*.
- In fact, the symbol $\int$ is actually a stylized capital S that is intended to signify the close connection between integration and summation.

- As given in the figure, for functions lying above the x

axis, the integral expressed by $I = \int_a^b f(x)\,dx$

corresponds to the area under the curve of *f(x)* between *x*= *a* and *b*.

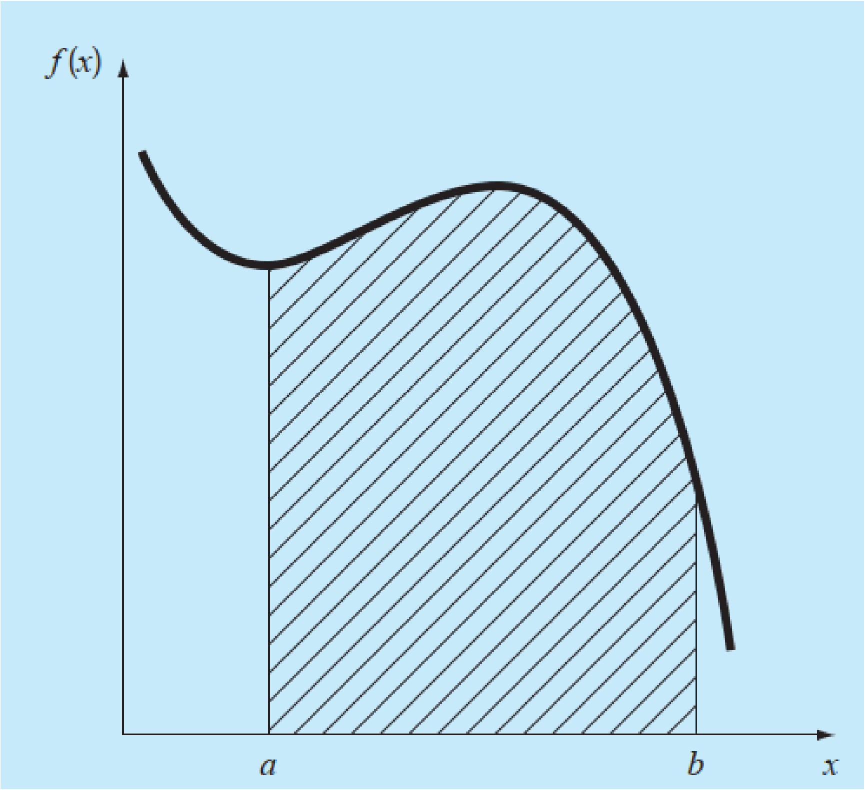


FIGURE 1 *Graphical representation of the integral of f(x) between the limits x = a to b. The integral is equivalent to the area under the curve.*

- As stated before, the integral and differentiation are inverse to each other.
- For example, if we are given a function *y(t)* that specifies an object's position as a function of time, differentiation provides a means to determine its velocity, as in Figure 2(*a*).

$$v(t) = \frac{d}{dt} y(t)$$

- Conversely, if we are provided with velocity as a function of time, integration can be used to determine its position, as in Figure 2(b).

$$y(t) = \int_0^t v(t)\, dt$$

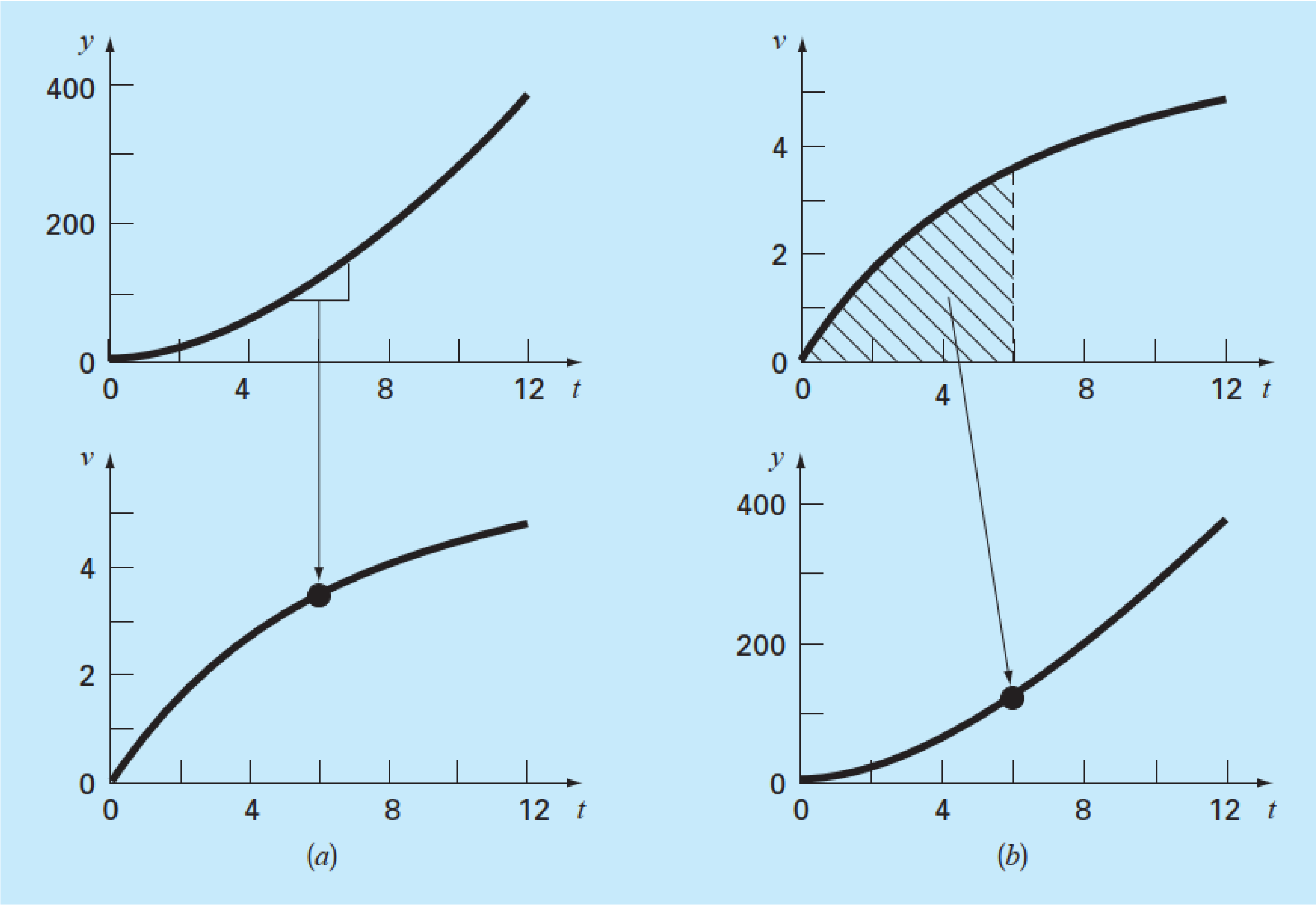


FIGURE 2 *The contrast between (a) differentiation and (b) integration.*

- Thus, we can make the general claim that the evaluation of the integral $I = \int_a^b f(x)\,dx$

is equivalent to solving the differential equation $\frac{dy}{dx} = f(x)$ for $y(b)$ given the initial condition $y(a) = 0$.

As mentioned in previous chapter, the function to be differentiated or integrated will typically be in one of the following three forms:

1. A simple continuous function such as a polynomial, an exponential, or a trigonometric function.

2. A complicated continuous function that is difficult or impossible to differentiate or integrate directly.

3. A tabulated function where values of x and $f(x)$ are given at a number of discrete points, as is often the case with experimental or field data.

- Visually oriented approaches were employed to integrate tabulated data and complicated functions in the precomputer era.
- A simple intuitive approach is to plot the function on a grid in Figure 3 and count the number of boxes that approximate the area.

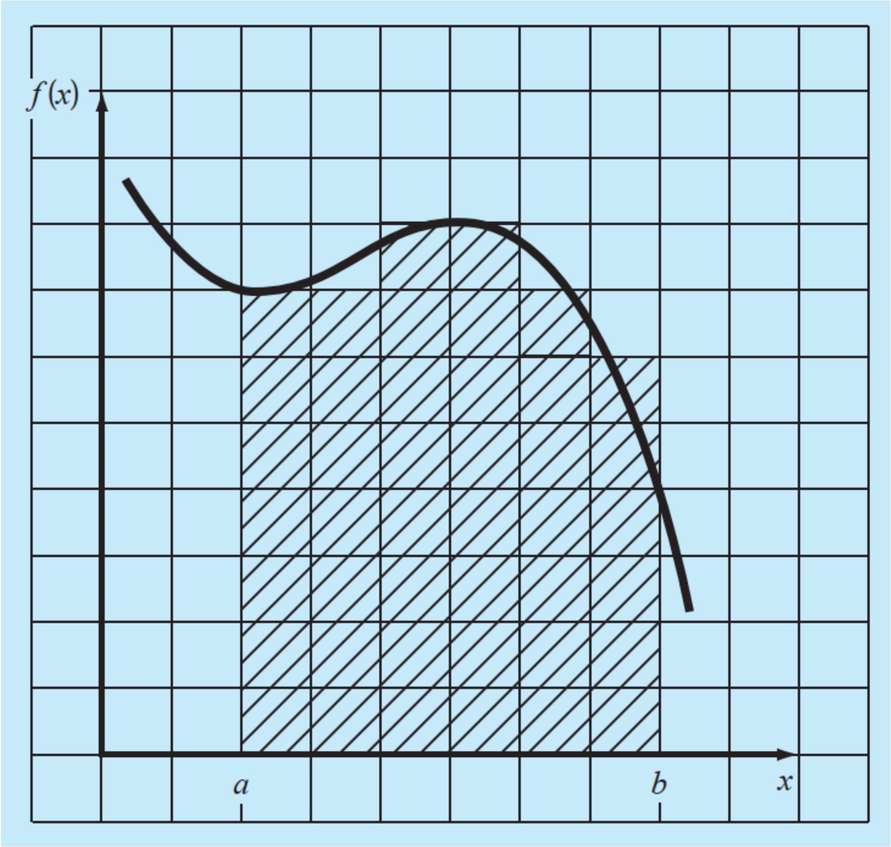


FIGURE 3 *The use of a grid to approximate an integral.*

- Another commonsense approach is to divide the area into vertical segments, or strips, with a height equal to the function value at the midpoint of each strip (Fig. 4). The area of the rectangles can then be calculated and summed to estimate the total area.
- In this approach, it is assumed that the value at the midpoint provides a valid approximation of the average height of the function for each strip. As with the grid method, refined estimates are possible by using more (and thinner) strips to approximate the integral.

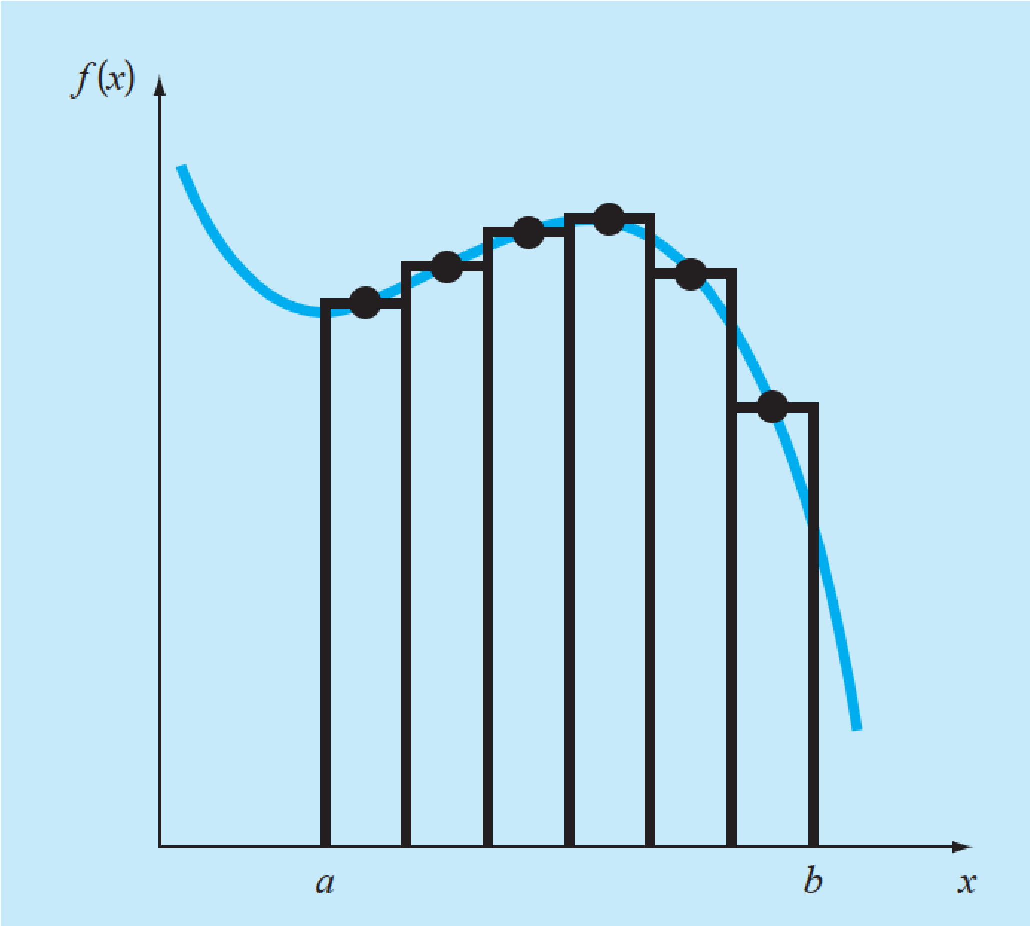


FIGURE 4 *The use of rectangles, or strips, to approximate the integral.*

10.2. MATHEMATICAL BACKGROUND

- General rules are available for integration of a function. To determine an integral between specified limits,

$$I = \int_a^b f(x)\,dx$$

- According to the *fundamental theorem* of integral calculus, equation above is evaluated as

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

where *F(x)* = the integral of *f(x)*- that is, any function such that $F'(x) = f(x)$. Nomenclature on the right side stands for

$$F(x)\Big|_a^b = F(b) - F(a)$$

Example:

$$I = \int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)\, dx$$

For this case, the function is a simple polynomial that can be integrated analytically by evaluating each term according to the rule

$$\int_a^b x^n\, dx = \left. \frac{x^{n+1}}{n+1} \right|_a^b$$

where n cannot equal -1. Applying this rule to each term in Eq.

$$I = 0.2x + 12.5x^2 - \frac{200}{3}x^3 + 168.75x^4 - 180x^5 + \left. \frac{400}{6}x^6 \right|_0^{0.8}$$

$$I = 1.6405333.$$

This value is equal to the area under the original polynomial between $x = 0$ and 0.8.

TABLE *Some simple integrals.*

$$\int u\,dv = uv - \int v\,du$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int a^{bx}\,dx = \frac{a^{bx}}{b \ln a} + C \qquad a > 0,\ a \neq 1$$

$$\int \frac{dx}{x} = \ln|x| + C \qquad x \neq 0$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \ln|x|\,dx = x\ln|x| - x + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{\sqrt{ab}}{a}x + C$$

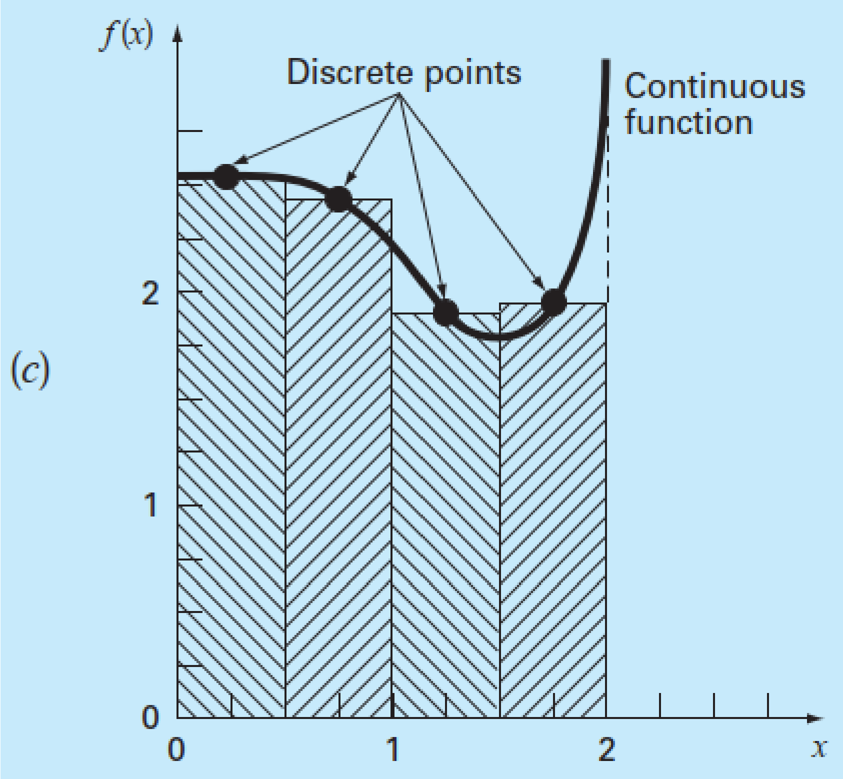
10.3. NUMERICAL INTEGRATION

- Although such simple approaches have utility for quick estimates, alternative numerical techniques are available for the same purpose.
- As in the simple strip method, numerical integration techniques utilize data at discrete points.
- Although continuous functions are not originally in discrete form, it is usually a simple proposition to use the given equation to generate a table of values.

(a) $$\int_0^2 \frac{2 + \cos(1 + x^{3/2})}{\sqrt{1 + 0.5 \sin x}} e^{0.5x}\, dx$$

(b)

x	$f(x)$
0.25	2.599
0.75	2.414
1.25	1.945
1.75	1.993



10.3.1 NEWTON-COTES INTEGRATION FORMULAS

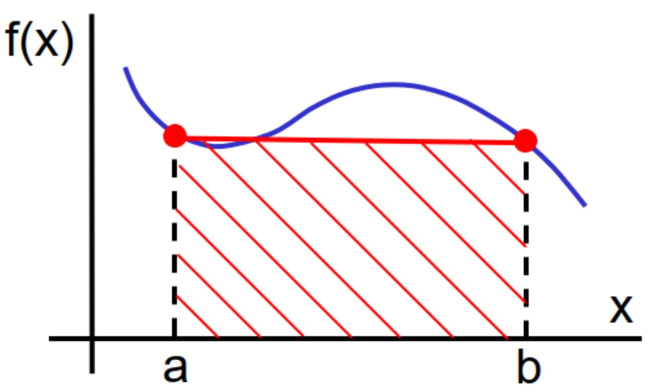
- **Idea:** Replace a complicated function or a tabulated data with an approximating (interpolating) function.

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

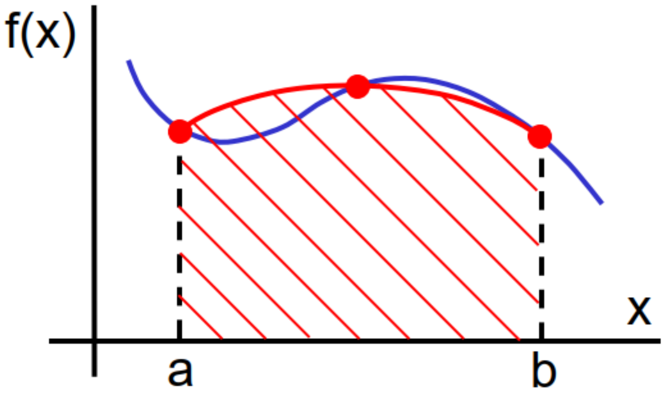
where $f_n(x)$ = a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

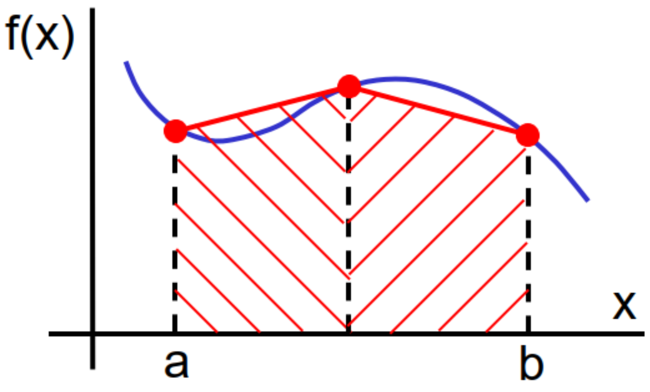
where n is the order of the polynomial.



1st order polynomial



2nd order polynomial

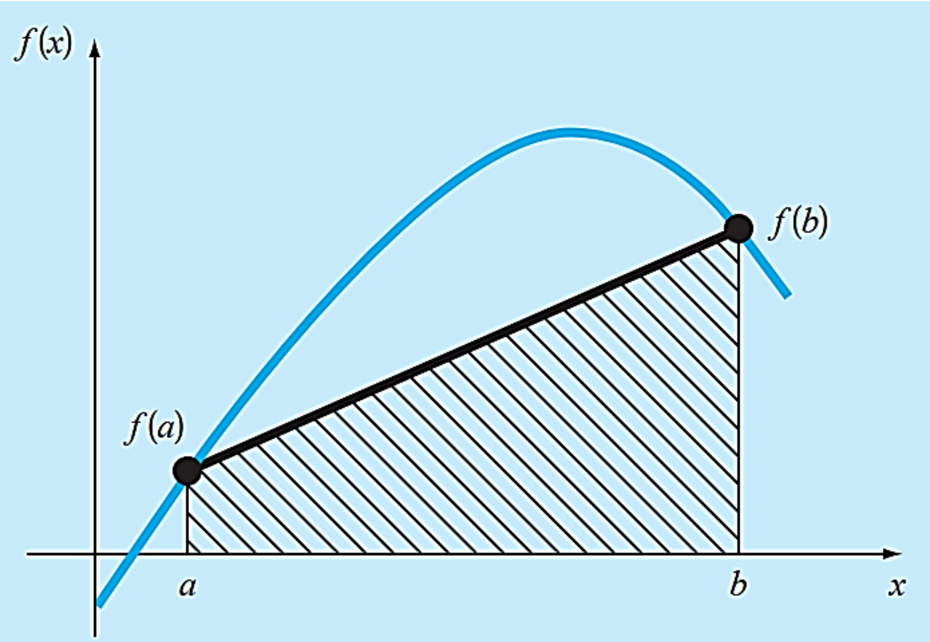


1st order with multiple segments

Trapezoidal Rule:

Idea: The trapezoidal rule is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial in previous equation is first-order:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$



- Recall from intepolation chapter that a straight line can be represented as

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

- The area under this straight line is an estimate of the integral of $f(x)$ between the limits a and b:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

The result of the integration

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

I= width average height*

Derivation of Trapezoidal Rule with Newton's Divided Difference Method

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

$$f_1(x) = \frac{f(b) - f(a)}{b - a}x + f(a) - \frac{af(b) - af(a)}{b - a}$$

Grouping the last two terms gives

$$f_1(x) = \frac{f(b) - f(a)}{b - a}x + \frac{bf(a) - af(a) - af(b) + af(a)}{b - a}$$

$$f_1(x) = \frac{f(b) - f(a)}{b - a}x + \frac{bf(a) - af(b)}{b - a}$$

which can be integrated between $x = a$ and $x = b$ to yield

$$I = \frac{f(b) - f(a)}{b - a}\frac{x^2}{2} + \frac{bf(a) - af(b)}{b - a}x\Big|_a^b$$

This result can be evaluated to give

$$I = \frac{f(b) - f(a)}{b - a}\frac{(b^2 - a^2)}{2} + \frac{bf(a) - af(b)}{b - a}(b - a)$$

Now, since $b^2 - a^2 = (b - a)(b + a)$,

$$I = [f(b) - f(a)]\frac{b + a}{2} + bf(a) - af(b)$$

Multiplying and collecting terms yields

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

which is the formula for the trapezoidal rule.

Derivation of Trapezoidal Rule with Newton-Gregory Method

Newton's formula, or the *Newton-Gregory forward formula,*

$$f_n(x) = f(x_0) + \Delta f(x_0)\alpha + \frac{\Delta^2 f(x_0)}{2!}\alpha(\alpha-1)$$

$$+\cdots+\frac{\Delta^n f(x_0)}{n!}\alpha(\alpha-1)\cdots(\alpha-n+1) \quad \text{where} \quad \alpha = \frac{x-x_0}{h}$$

$$I = \int_a^b \left[\underbrace{f(a) + \Delta f(a)\alpha}_{\text{1st order N.G formula}} + \underbrace{\frac{f''(\xi)}{2}\alpha(\alpha-1)h^2}_{\text{Remainder term}}\right] dx$$

Change integration limits from x to α

$$\alpha = (x-a)/h, \qquad dx = h\, d\alpha \qquad \int_{x=q}^{x=b} dx = \int_{\alpha=0}^{\alpha=1} h d\alpha$$

$$I = h\int_0^1 \left[f(a) + \Delta f(a)\alpha + \frac{f''(\xi)}{2}\alpha(\alpha-1)h^2\right] d\alpha$$

If it is assumed that, for small h, the term $f''(\xi)$ is approximately constant, this equation can be integrated:

$$I = h\left[\alpha f(a) + \frac{\alpha^2}{2}\Delta f(a) + \left(\frac{\alpha^3}{6} - \frac{\alpha^2}{4}\right) f''(\xi)h^2\right]_0^1$$

and evaluated as

$$I = h\left[f(a) + \frac{\Delta f(a)}{2}\right] - \frac{1}{12}f''(\xi)h^3$$

Because $\Delta f(a) = f(b) - f(a)$, the result can be written as

$$I = \underbrace{h\frac{f(a)+f(b)}{2}}_{\text{Trapezoidal rule}} \underbrace{- \frac{1}{12}f''(\xi)h^3}_{\text{Truncation error}}$$

Error in Trapezoidal Rule:

When we employ the integral under a straight-line segment to approximate the integral under a curve, we obviously can incur an error that may be substantial.

$$I = h\frac{f(a) + f(b)}{2} - \frac{1}{12}f''(\xi)h^3$$

- Usually $f''(\xi)$ in the term can not be evaluated, since ξ is not known.
- If the function f is known than $f''(\xi)$ can be approximated with an average 2nd derivative.

$$f''(\xi) \approx \bar{f}''(x) = \frac{\int_a^b f''(x)dx}{b-a} \rightarrow \quad \boxed{E_a = -\frac{h^3}{12}\bar{f}''(x)}$$

Example:

Numerically integrate the following polynomial from $a=0$ to $b=0.8$. (Exact value is *1.640533*)

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

the *trapezoidal rule*

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

The function values

$$f(0) = 0.2$$
$$f(0.8) = 0.232$$

substituted into Eq. $\quad I \cong 0.8\frac{0.2+0.232}{2} = 0.1728$

which represents an error of

$E_t = 1.640533 - 0.1728 = 1.467733$ a percent relative error of $\varepsilon_t = 89.5\%$.

In actual situations, we would have no foreknowledge of the true value. Therefore, an approximate error estimate is required.

$$f''(\xi) \approx \bar{f}''(x) = \frac{\int_a^b f''(x)dx}{b-a} \rightarrow \quad E_a = -\frac{h^3}{12}\bar{f}''(x)$$

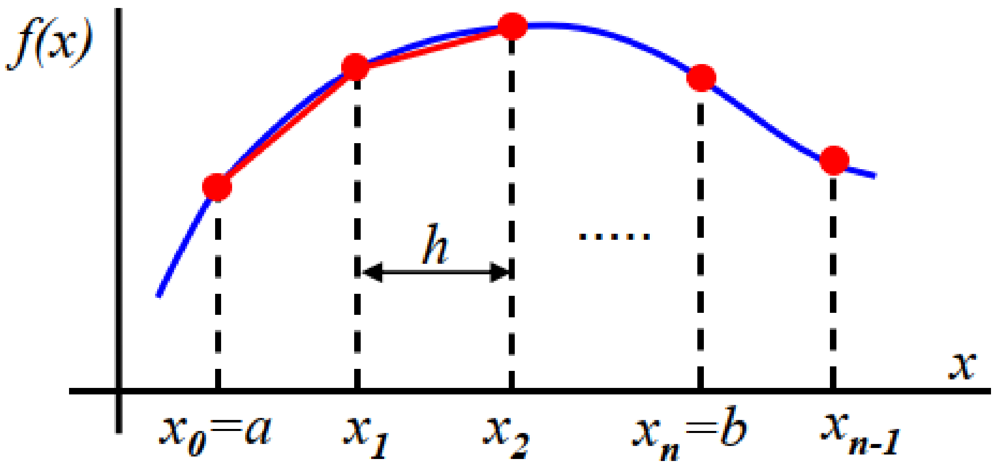
$$f''(x) = -400 + 4050x - 10{,}800x^2 + 8000x^3$$

The average value of the second derivative can be computed

$$\bar{f}''(x) = \frac{\int_0^{0.8} (-400 + 4050x - 10{,}800x^2 + 8000x^3)\,dx}{0.8 - 0} = -60$$

$$E_a = -\frac{1}{12}(-60)(0.8)^3 = 2.56$$

The Multiple-Application Trapezoidal Rule



- In general, we have *n+1* points and *n* intervals (segments).
- If the points are equispaced $h = (b\text{-}a)/n$

$$I = \int_a^b f(x)\,dx = \int_{x0}^{x1} f(x)\,dx + \int_{x1}^{x2} f(x)\,dx + \ldots + \int_{x(n-1)}^{xn} f(x)\,dx$$

$$I \approx h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \ldots + h\frac{f(x_{n-1})+f(x_n)}{2}$$

$$I \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right] = (b-a)\left[\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}\right]$$

Error in Multiple-Application Trapezoidal Rule:

- Add the individual errors for each interval

$$E_t = -\frac{h^3}{12}\sum_{i=1}^{n} f''(\xi_i) = -\frac{(b-a)^3}{12\,n^3}\sum_{i=1}^{n} f''(\xi_i)$$

- Use a single ξ for the entire interval

$$\sum_{i=1}^{n} f''(\xi_i) = n\, f''(\xi)$$

$$E_t = -\frac{(b-a)^3}{12\,n^2}\, f''(\xi) = -\frac{(b-a)\,h^2}{12}\, f''(\xi)$$

- Similar to the single application of the trapezoidal rule, if the function f is known than $f''(\xi)$ can be approximated with an average 2nd derivative

$$f''(\xi) \approx \bar{f}''(x) = \frac{\int_a^b f''(x)\,dx}{b-a} \quad \rightarrow \quad E_a = -\frac{(b-a)^3}{12\,n^2}\,\bar{f}''(x)$$

Example:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

the *trapezoidal rule*

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

The function values

$$f(0) = 0.2$$
$$f(0.8) = 0.232$$

substituted into Eq. $$I \cong 0.8\frac{0.2 + 0.232}{2} = 0.1728$$

which represents an error of

$E_t = 1.640533 - 0.1728 = 1.467733$ a percent relative error of $\varepsilon_t = 89.5\%$.

Example: Use the two-segment trapezoidal rule to estimate the integral of from $a = 0$ to $b = 0.8$.

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ the correct value for the integral is 1.640533.

Solution.

$n = 2$ $(h = 0.4)$:

$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$E_t = 1.640533 - 1.0688 = 0.57173 \qquad \varepsilon_t = 34.9\%$

$$E_a = -\frac{0.8^3}{12(2)^2}(-60) = 0.64$$

Example: Integrate f(x)= e^x from a=1.5 to a=2.5 using the Trapezoidal Rule. Estimate the error.

True value is $e^{2.5} – e^{1.5} = 7.700805$

Using the trapezoidal rule:

$a = 1.5,\quad b = 2.5,\quad h = 2.5 – 1.5 = 1.0$

$$I \approx h\,\frac{f(a)+f(b)}{2} = 1.0\,\frac{e^{1.5}+e^{2.5}}{2} = 8.332092$$

$E_t = - 0.631287,\quad \varepsilon_t = - 8.2\ \%$

Estimated error: $$E_a \approx -\frac{h^3}{12}\,\frac{\int_a^b f''(x)\,dx}{b-a} = -\frac{1^3}{12}\,\frac{\int_{1.5}^{2.5} e^x\,dx}{1.0} = -0.641734$$

Example: Integrate $f(x)= e^x$ from a=1.5 to a=2.5 using the Trapezoidal Rule. Use a step size of 0.25. True value of the integral is 7.700805.

$a = 1.5, \quad b = 2.5, \quad h = 0.25$ than we have n=4 intervals.

$$\mathbf{I} \approx (b-a)\left[\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_1)}{2n}\right] = (2.5-1.5)\left[\frac{e^{1.5} + 2\,(e^{1.75} + e^{2} + e^{2.25}) + e^{2.5}}{2\,(4)}\right]$$

$I \approx 7.740872$, $\quad E_t = -\,0.040067, \quad \varepsilon_t = -\,0.5\ \%$

Estimated error: $$E_a = -\frac{(b-a)^3}{12\,n^2}\,\overline{f''(x)} = -\frac{(1.0)^3}{12\,(4)^2}\,\frac{\int_{1.5}^{2.5} f''(x)\,dx}{(1.0)} = -\,0.040108$$

Note that the calculation of E_a requires the evaluation of the same integral that the question asks for. Of course the integral in E_a can laso be calculated numerically. This approach also gives

$$E_a = -\frac{(1.0)^3}{12\,(4)^2}\,\frac{7.740872}{(1.0)} = -\,0.040317$$

Simpson's 1/3 Rule:

Aside from applying the trapezoidal rule with finer segmentation, another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points.

Idea: Simpson's 1/3 rule results when a second-order interpolating polynomial is used to approximate the integration.

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second-order Lagrange polynomial

, the integral becomes

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

After integration and algebraic manipulation, the following formula results:

$$I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \quad \text{where, for this case, } h = (b - a)/2.$$

Simpson's 1/3 rule can also be expressed

$$I \cong \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}}$$

Derivation of Simpson's 1/3 Rule with Newton-Gregory Method

Newton's formula, or the *Newton-Gregory forward formula*,

$$I = \int_{x_0}^{x_2} \left[f(x_0) + \Delta f(x_0)\alpha + \frac{\Delta^2 f(x_0)}{2}\alpha(\alpha-1) + \frac{\Delta^3 f(x_0)}{6}\alpha(\alpha-1)(\alpha-2) + \frac{f^{(4)}(\xi)}{24}\alpha(\alpha-1)(\alpha-2)(\alpha-3)h^4 \right] dx$$

Notice that we have written the polynomial up to the fourth-order term rather than the third-order term as would be expected.

The integral is from $\alpha = 0$ to 2:

$$I = h\int_0^2 \left[f(x_0) + \Delta f(x_0)\alpha + \frac{\Delta^2 f(x_0)}{2}\alpha(\alpha-1) + \frac{\Delta^3 f(x_0)}{6}\alpha(\alpha-1)(\alpha-2) + \frac{f^{(4)}(\xi)}{24}\alpha(\alpha-1)(\alpha-2)(\alpha-3)h^4 \right] d\alpha$$

$$I = h\left[\alpha f(x_0) + \frac{\alpha^2}{2}\Delta f(x_0) + \left(\frac{\alpha^3}{6} - \frac{\alpha^2}{4}\right)\Delta^2 f(x_0) + \left(\frac{\alpha^4}{24} - \frac{\alpha^3}{6} + \frac{\alpha^2}{6}\right)\Delta^3 f(x_0) + \left(\frac{\alpha^5}{120} - \frac{\alpha^4}{16} + \frac{11\alpha^3}{72} - \frac{\alpha^2}{8}\right) f^{(4)}(\xi)h^4 \right]_0^2$$

and evaluated for the limits to give

$$I = h\left[2f(x_0) + 2\Delta f(x_0) + \frac{\Delta^2 f(x_0)}{3} + (0)\Delta^3 f(x_0) - \frac{1}{90} f^{(4)}(\xi)h^4 \right]$$

$$I = \underbrace{\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}_{\text{Simpson's 1/3 rule}} - \underbrace{\frac{1}{90} f^{(4)}(\xi)h^5}_{\text{Truncation error}}$$

Error in Simpson's 1/3 Rule:

As given in the previous part, Simpson's 1/3 rule has a truncation error of

$$E_t = -\frac{1}{90} h^5 f^{(4)}(\xi)$$

or, because $h = (b - a)/2$,

$$\boxed{E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)}$$

where ξ lies somewhere in the interval from a to b.

Thus, Simpson's 1/3 rule is more accurate than the trapezoidal rule.

Example: Use the Simpson's 1/3 rule to estimate the integral from $a = 0$ to $b = 0.8$.

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ the correct value for the integral is 1.640533.

Solution.

$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$

$$I \cong 0.8\frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

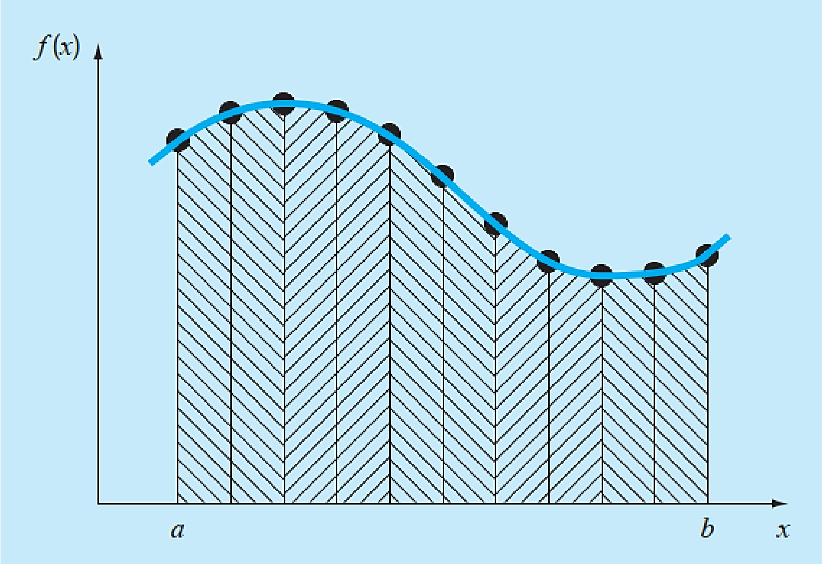
which represents an exact error of

$$E_t = 1.640533 - 1.367467 = 0.2730667 \qquad \varepsilon_t = 16.6\%$$

which is approximately 5 times more accurate than for a single application of the trapezoidal rule

The estimated error is $E_a = -\dfrac{(0.8)^5}{2880}(-2400) = 0.2730667$

The Multiple-Application Simpson's 1/3 Rule



Just as with the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width as given in figure

$$h = \frac{b-a}{n}$$

The total integral can be represented as

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

Substituting Simpson's 1/3 rule for the individual integral yields

$$I \cong 2h\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \cdots + 2h\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

or, combining terms

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum\limits_{i=1,3,5}^{n-1} f(x_i) + 2\sum\limits_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$

Error in Multiple-Application Simpson's 1/3 Rule:

An error estimate for the multiple-application Simpson's rule is obtained in the same fashion as for the trapezoidal rule by summing the individual errors for the segments and averaging the derivative to yield

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

where $\bar{f}^{(4)}$ is the average fourth derivative for the interval.

Example: Use the multi-application Simpson's 1/3 rule to estimate the integral from $a = 0$ to $b = 0.8$.

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ the correct value for the integral is 1.640533.

Solution. $n = 4$ $(h = 0.2)$:

$$f(0) = 0.2 \qquad f(0.2) = 1.288$$
$$f(0.4) = 2.456 \qquad f(0.6) = 3.464$$
$$f(0.8) = 0.232$$

$$I = 0.8\,\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

$$E_t = 1.640533 - 1.623467 = 0.017067 \qquad \varepsilon_t = 1.04\%$$

The estimated error

$$E_a = -\frac{(0.8)^5}{180(4)^4}(-2400) = 0.017067$$

Summary on Newton-Cotes Integration Methods

TABLE 21.2 Newton-Cotes closed integration formulas. The formulas are presented in the format of Eq. (21.5) so that the weighting of the data points to estimate the average height is apparent. The step size is given by $h = (b - a)/n$.

Segments (n)	Points	Name	Formula	Truncation Error
1	2	Trapezoidal rule	$(b-a)\dfrac{f(x_0)+f(x_1)}{2}$	$-(1/12)h^3f''(\xi)$
2	3	Simpson's 1/3 rule	$(b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$	$-(1/90)h^5f^{(4)}(\xi)$
3	4	Simpson's 3/8 rule	$(b-a)\dfrac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$	$-(3/80)h^5f^{(4)}(\xi)$
4	5	Boole's rule	$(b-a)\dfrac{7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)}{90}$	$-(8/945)h^7f^{(6)}(\xi)$
5	6		$(b-a)\dfrac{19f(x_0)+75f(x_1)+50f(x_2)+50f(x_3)+75f(x_4)+19f(x_5)}{288}$	$-(275/12{,}096)h^7f^{(6)}(\xi)$

NEXT WEEK
NUMERICAL INTEGRATION