Ch.7. Transformation of Stress

Transformation of Plane Stress and Mohr’s Circle

Objectives

Application of stress transformation equations to plane stress situations to determine any stress component at a point.

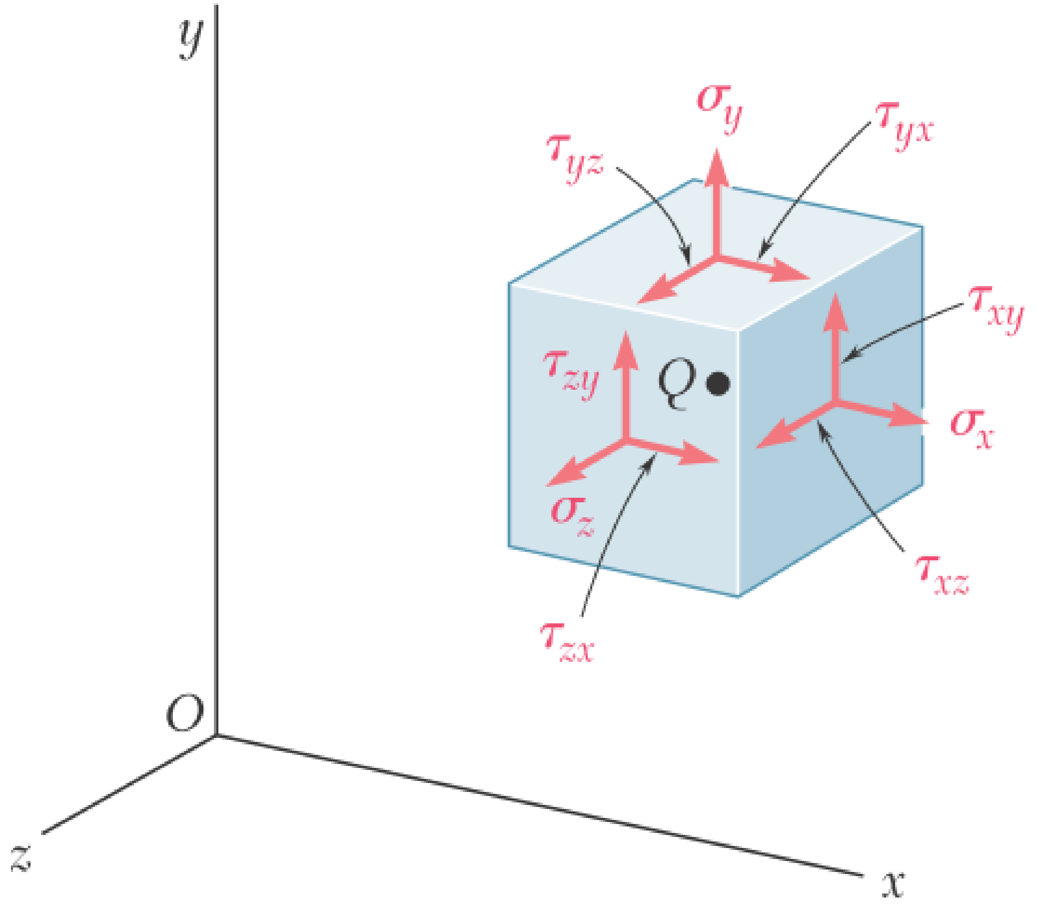
Application of the alternative Mohr's circle approach to perform plane stress transformations.

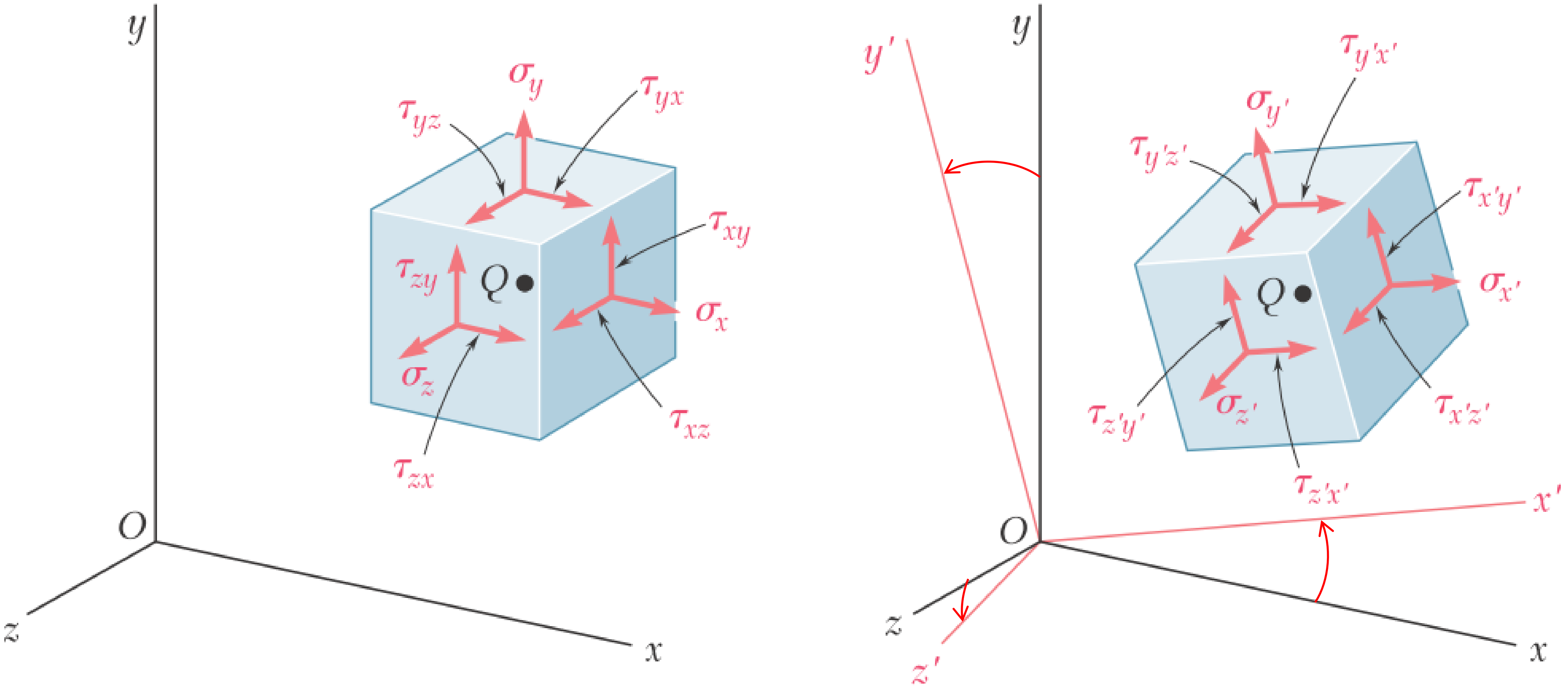
Usage of transformation techniques to identify key components of stress, such as principal stresses.

- The most general state of stress at a given point Q is represented by six components. Three of these components,

✓ σ_x, σ_y, and σ_z, are the normal stresses exerted on the faces of a small cubic element centered at Q with the same orientation as the coordinate axes.

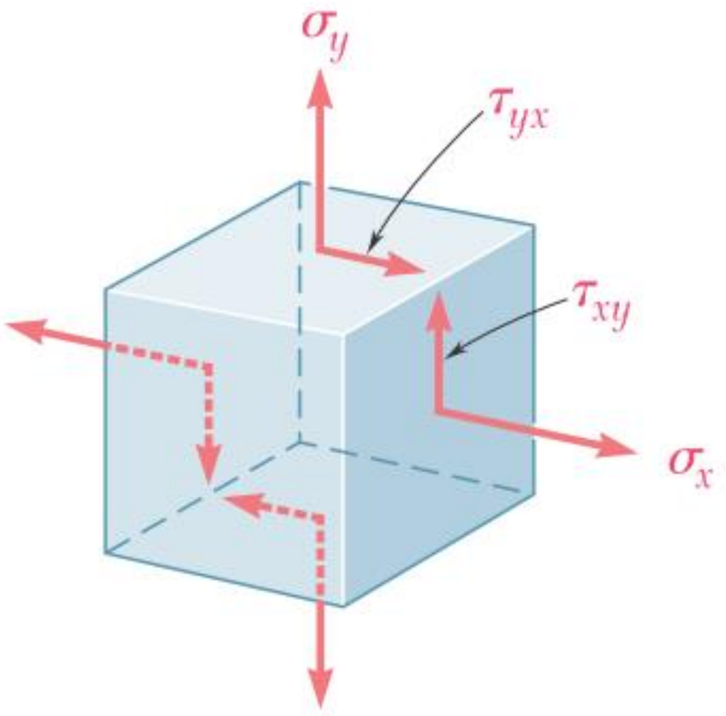
✓ The other three, τ_{xy}, τ_{yz}, and τ_{zx}, are the components of the shearing stresses on the same element.





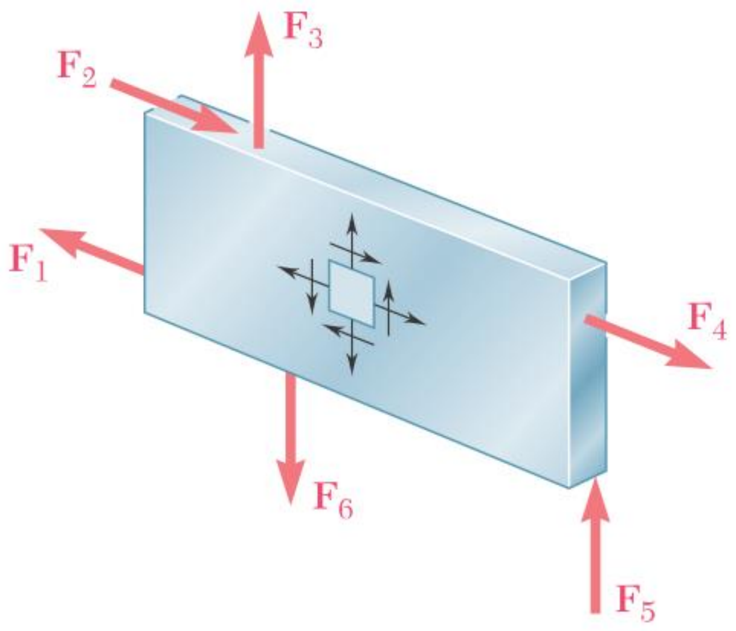
- The same state of stress can be represented by a different set of components if the coordinate axes are rotated

- Our discussion of the transformation of stress will deal mainly with ***plane stress***, i.e., with a situation in which two of the faces of the cubic element are free of any stress.
- If the z axis is chosen perpendicular to these faces, $\boldsymbol{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$, and the only remaining stress components are $\boldsymbol{\sigma_x}$**,** $\boldsymbol{\sigma_y}$**, and** $\boldsymbol{\tau_{xy}}$.

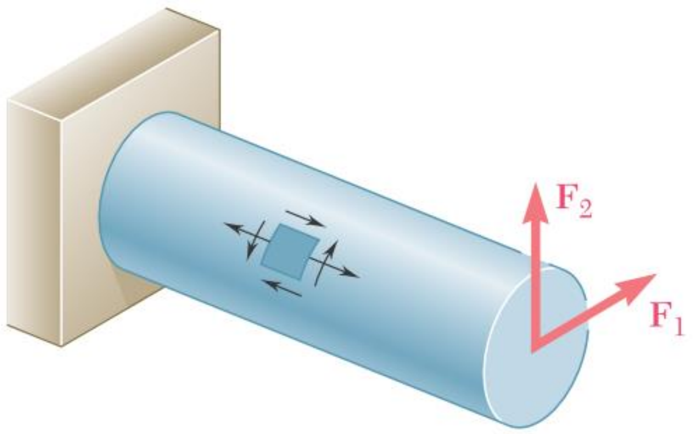


Plane stress representation

- The ***plane-stress*** situation occurs in a thin plate subjected to forces acting in the midplane of the plate.



- It also occurs on the free surface of a structural element or machine component where any point of the surface of that element or component is not subjected to an external force.

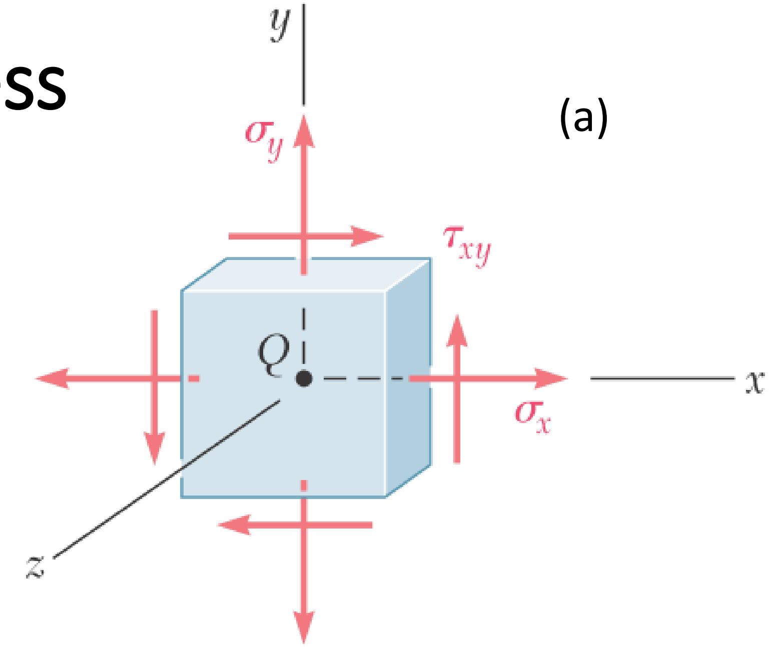


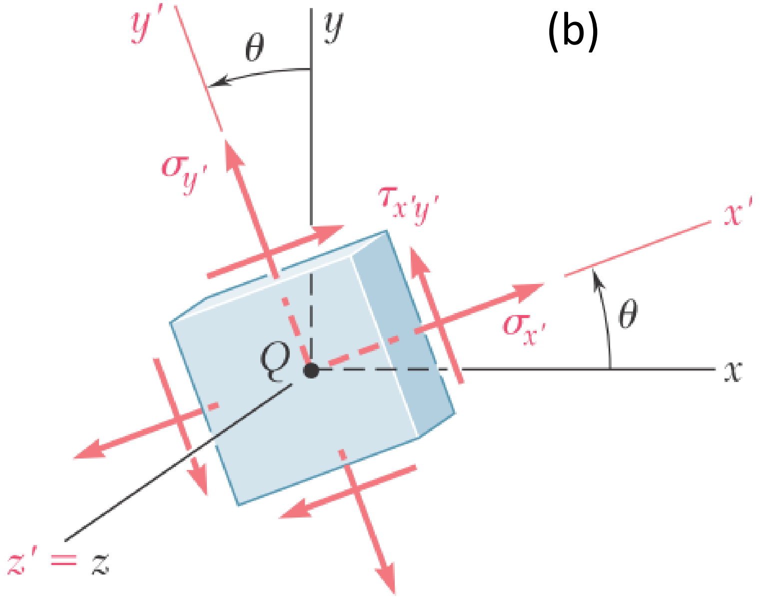
Transformation of Plane Stress

Transformation Equations

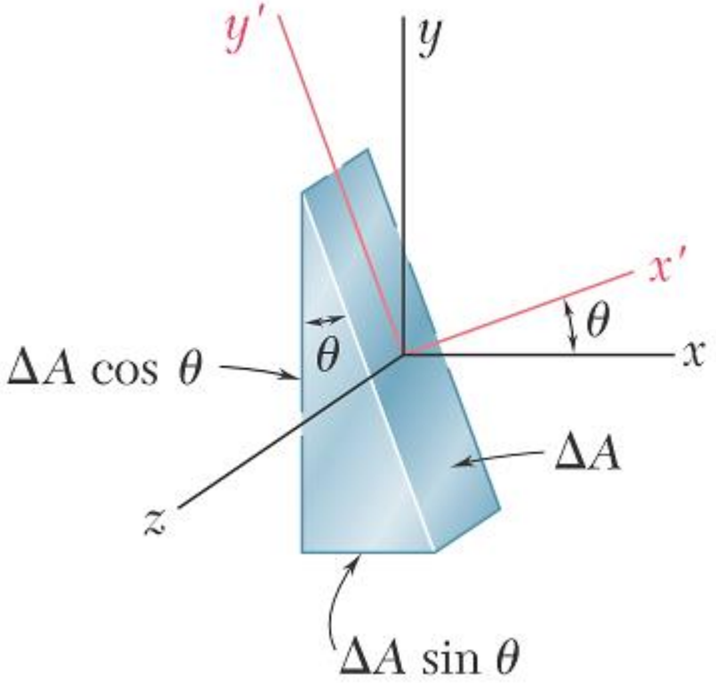
- Assume that a state of plane stress exists at point Q (with $\sigma_z = \tau_{zx} = \tau_{zy} = 0$) and is defined by the stress components σ_x, σ_y, and τ_{xy} associated with the element shown in Figure a.

- The stress components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with the element are determined after it has been rotated through an angle θ about the z axis. These components are given in terms of σ_x, σ_y, τ_{xy}, and θ (Figure b).

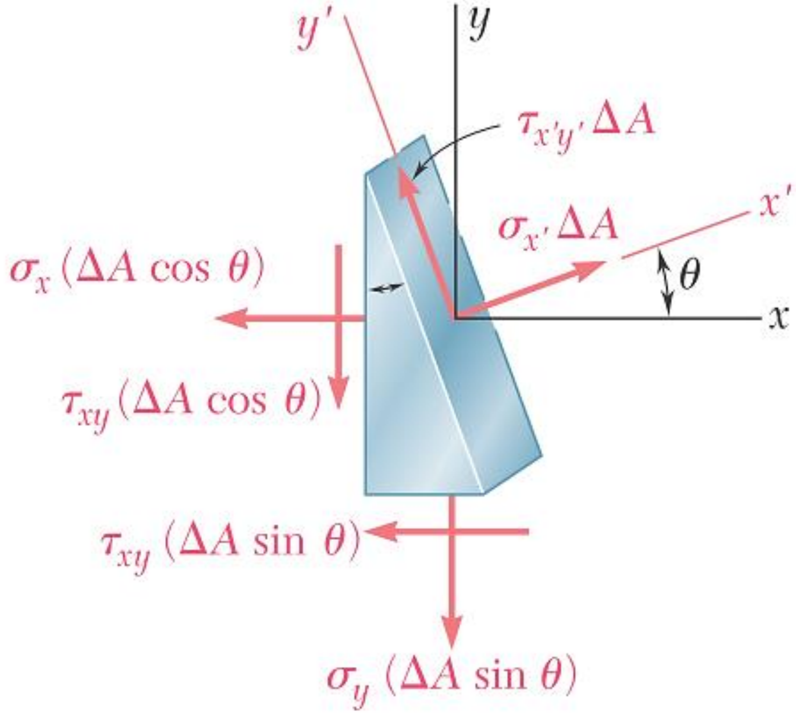




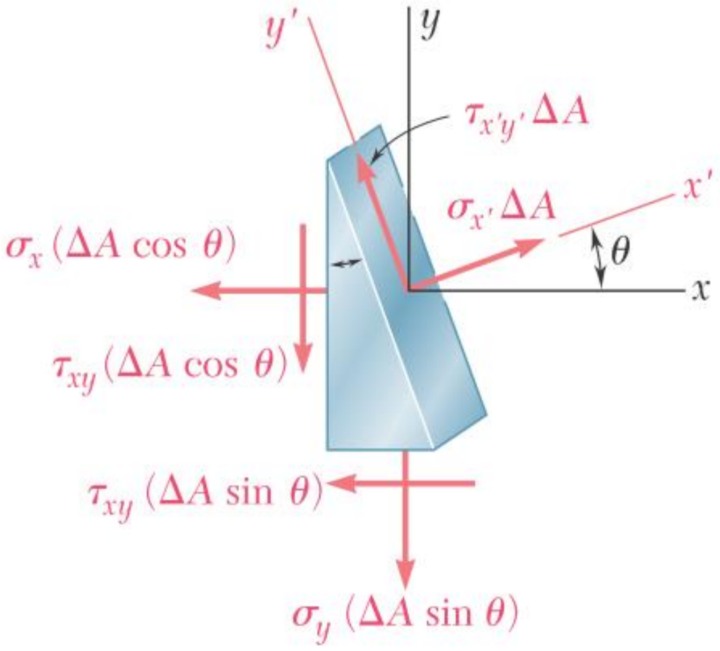
- In order to determine the normal stress $\sigma_{x'}$ and shearing stress $\tau_{x'y'}$ exerted on the face perpendicular to the x' axis, consider a prismatic element with faces perpendicular to the x, y, and x' axes.



Geometry of the element



Free-body diagram



- Using components along the x' and y' axes, the equilibrium equations are

$$\sum F_{x'} = 0: \quad \sigma_{x'}\,\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta - \sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

$$\sum F_{y'} = 0: \quad \tau_{x'y'}\,\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta - \sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

- Solving the first equation for $\sigma_{x'}$ and the second for $\tau_{x'y'}$:

$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Since

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \text{and} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad ☆$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

- The normal stress $\sigma_{y'}$ is obtained by replacing θ in ☆ equation by the angle $\theta + 90°$ that the y' axis forms with the x axis. Since $\cos(2\theta + 180°) = -cos2\theta$ and $\sin(2\theta + 180°) = -sin2\theta$,

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad ☆☆$$

- Adding ☆ and ☆☆ equations side by side:

$$\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$$

- Since $\sigma_z = \sigma_{z'} = 0$, we thus verify for plane stress that the sum of the normal stresses exerted on a cubic element of material is independent of the orientation of that element.

Principal Stresses and Maximum Shearing Stress

- Previous equations are the parametric equations of a circle. This means that,

if a set of rectangular axes is used to plot a point M of abscissa $\sigma_{x'}$ and ordinate $\tau_{x'y'}$ for any given parameter θ, all of the points obtained will lie on a circle.

- To establish this property, we eliminate θ from equations by first transposing $(\sigma_x + \sigma_y)/2$ in

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

- and squaring both members of the equation, then squaring both members of

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

- and finally adding member to member the two equations obtained above:

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

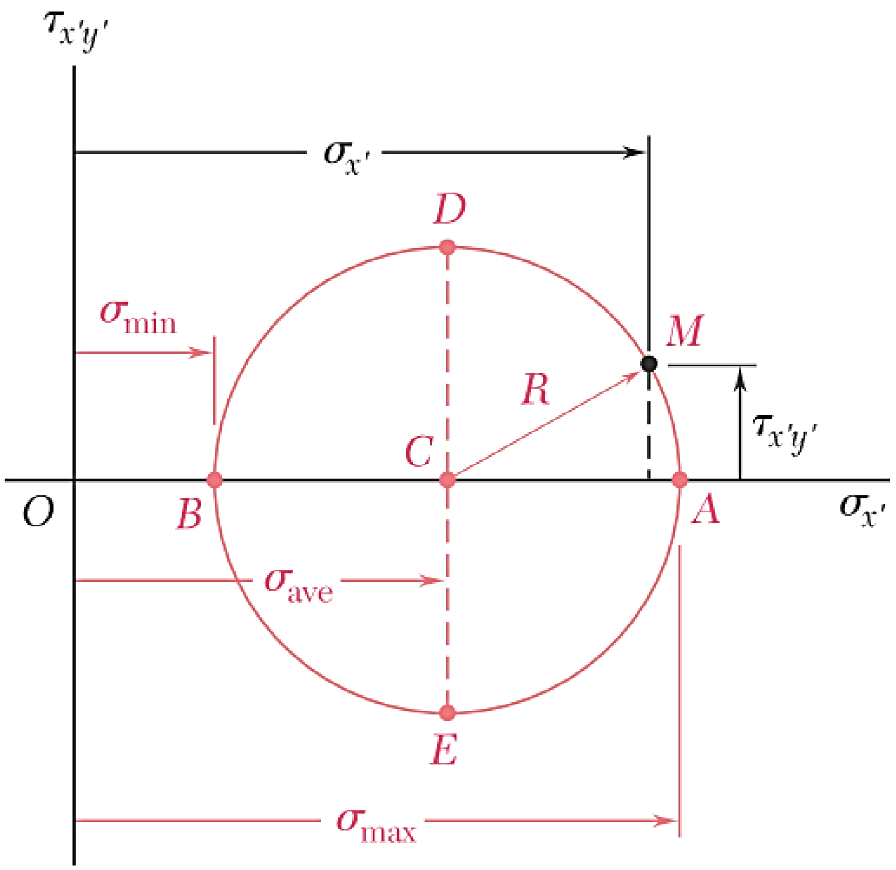
- Setting

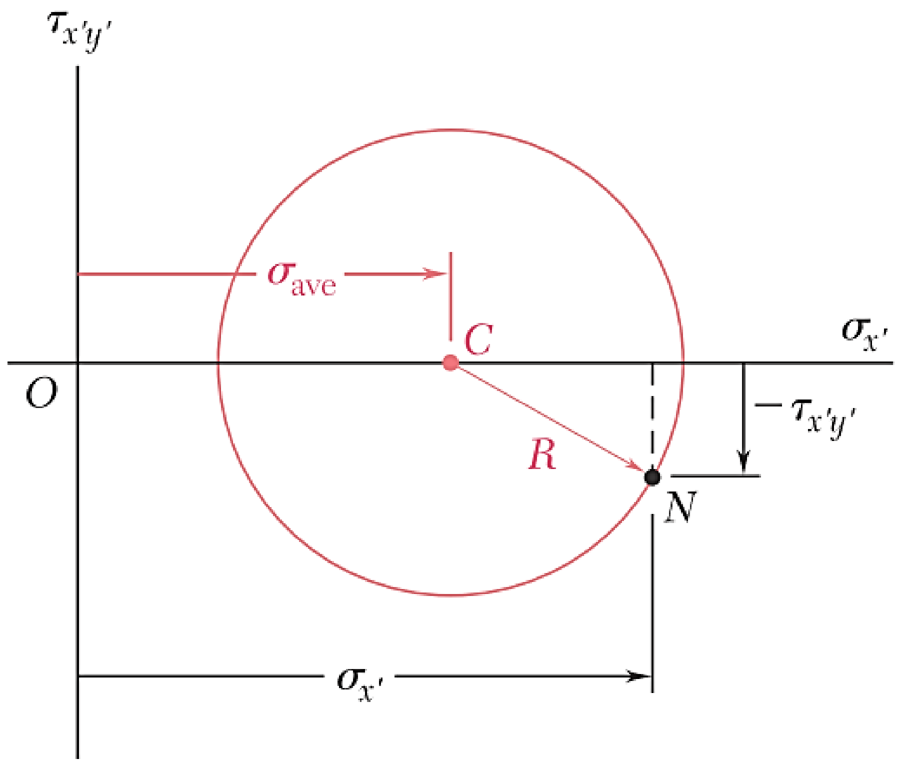
$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} \qquad \text{and} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- is given as

$$(\sigma_{x'} - \sigma_{\text{ave}})^2 + \tau_{x'y'}^2 = R^2$$

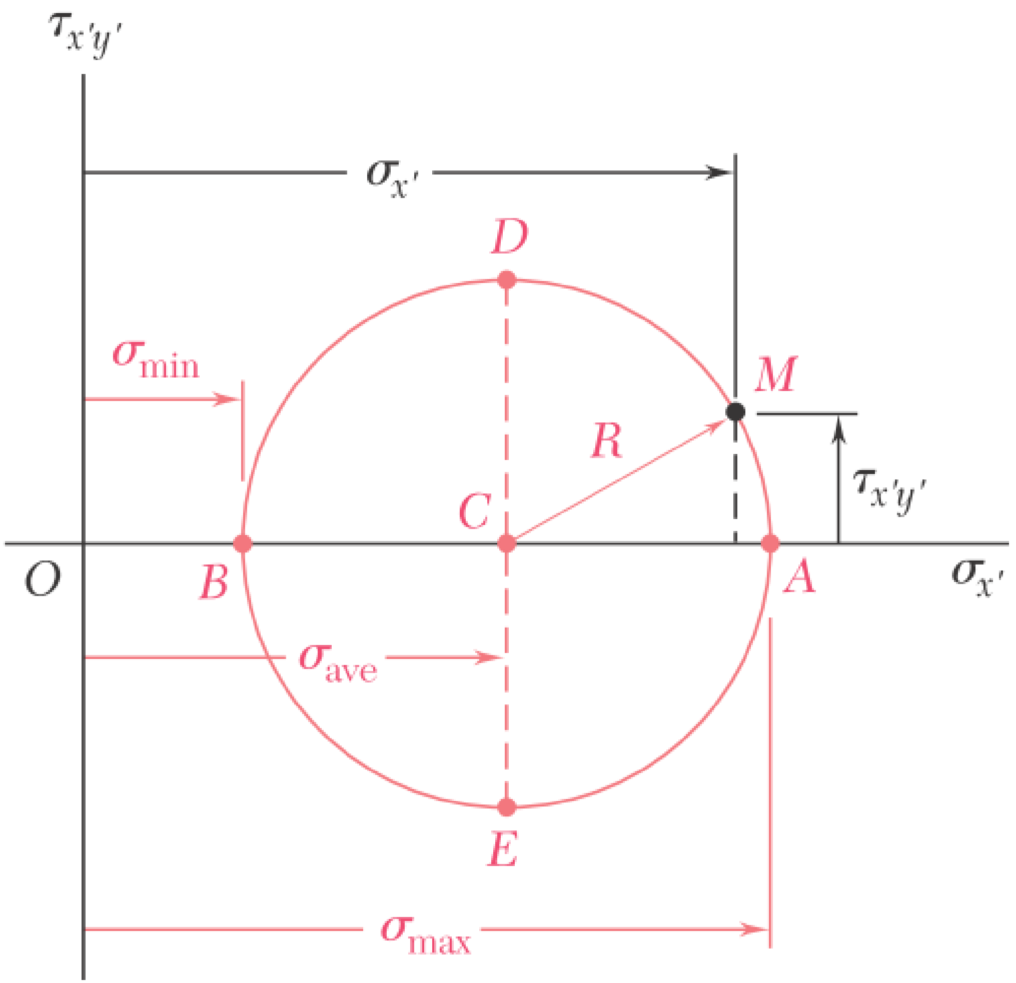
- which is the equation of a circle of radius R centered at the point C of abscissa σ_{ave} and ordinate O.





- Due to the symmetry of the circle about the horizontal axis, the same result is obtained if a point N of abscissa $\sigma_{x'}$ and ordinate $-\tau_{x'y'}$ is plotted instead of M.

- The points A and B where the circle intersects the horizontal axis are of special interest: point A corresponds to the maximum value of the normal stress $\sigma_{x'}$, while point B corresponds to its minimum value. Both points also correspond to a zero value of the shearing stress $\tau_{x'y'}$. Thus, the values θ_p of the parameter θ which correspond to points A and B can be obtained by setting $\tau_{x'y'} = 0$ in

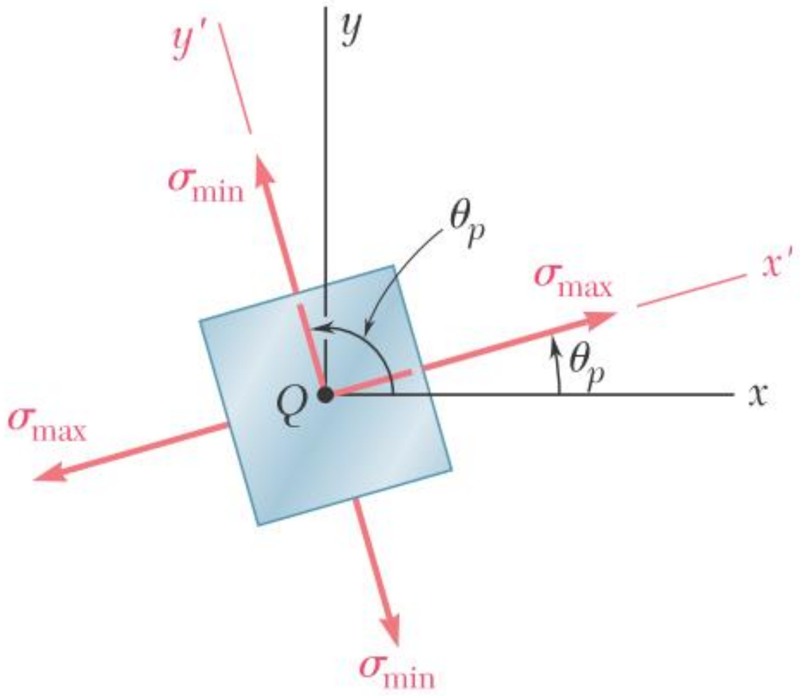


$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \Rightarrow \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

(orientation of the corresponding element)

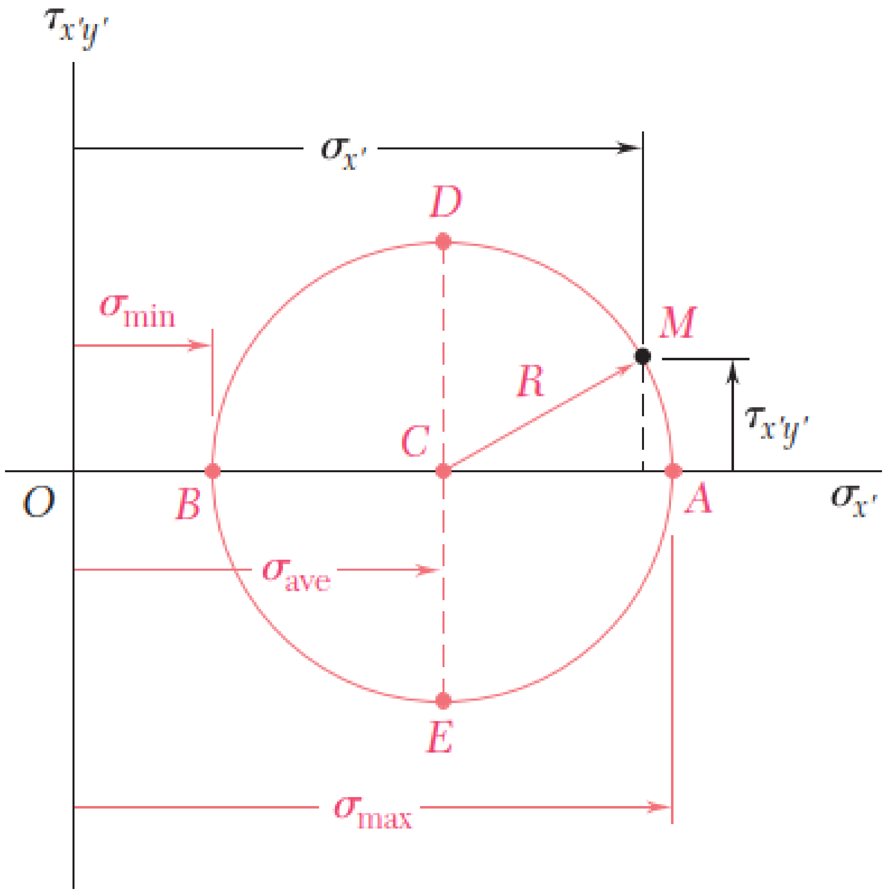
- This equation defines two values $2\theta_p$ that are $180°$ apart and thus two values θ_p that are $90°$ apart. Either value can be used to determine the orientation of the corresponding element.



- The planes containing the faces of the element obtained in this way are the principal planes of stress at point Q, and the corresponding values σ_{max} and σ_{min} exerted on these planes are the **principal stresses** at Q.
- Since both values θ_p defined by the equation below are obtained by setting $\tau_{x'y'} = 0$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- It is clear that no shearing stress is exerted on the principal planes.



From the figure given:

$$\sigma_{max} = \sigma_{ave} + R \quad \text{and} \quad \sigma_{min} = \sigma_{ave} - R$$

Substituting σ_{ave} and R:

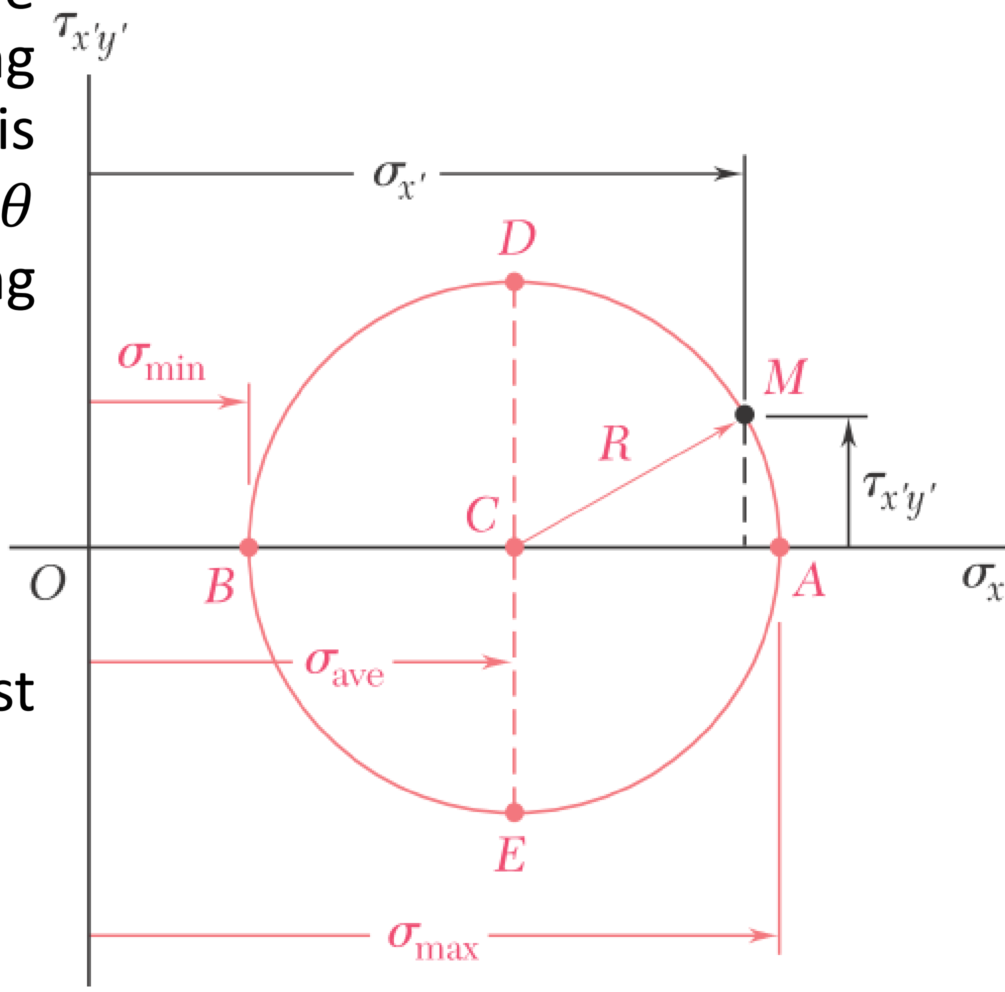
$$\sigma_{max,\,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Points D and E located on the vertical diameter of the circle correspond to the largest value of the shearing stress $\tau_{x'y'}$. Since the abscissa of points D and E is $\sigma_{ave} = (\sigma_x + \sigma_y)/2$, the values θ_s of the parameter θ corresponding to these points are obtained by setting $\sigma_{x'} = (\sigma_x + \sigma_y)/2$ in

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

- The sum of the last two terms in that equation must be zero. Thus, for $\theta = \theta_s$,

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$



$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

- This equation defines two values $2\theta_s$ that are $180°$ apart, and thus two values θ_s that are $90°$ apart. Either of these values can be used to determine the orientation of the element corresponding to the maximum shearing stress.
- It is also shown that the *maximum value of the shearing stress* is equal to the *radius R of the circle*.

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$