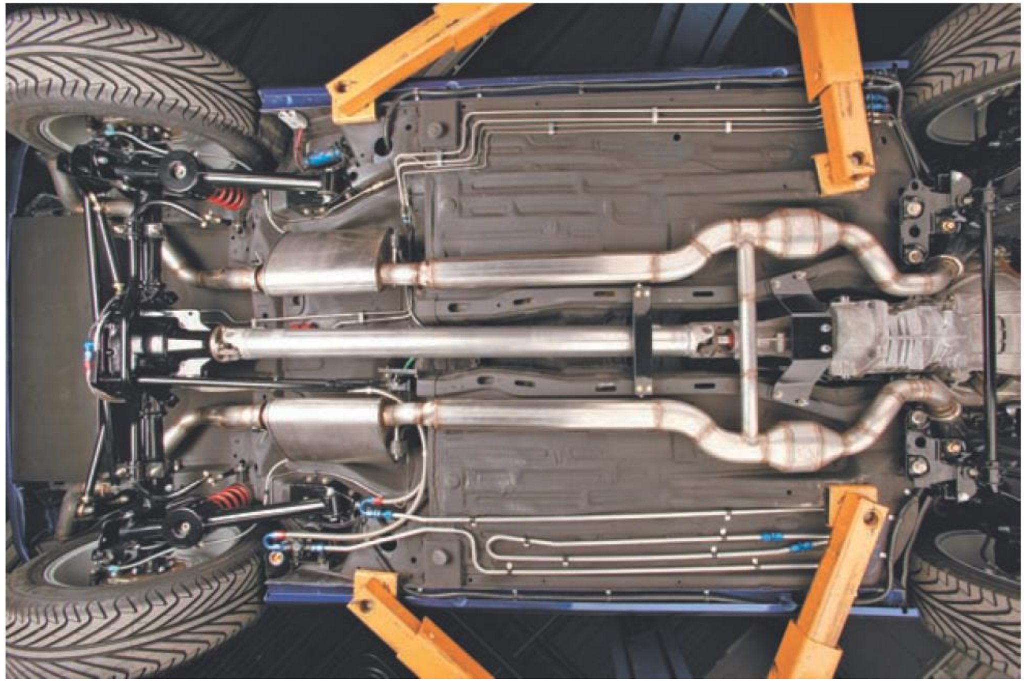
CH.3 TORSION

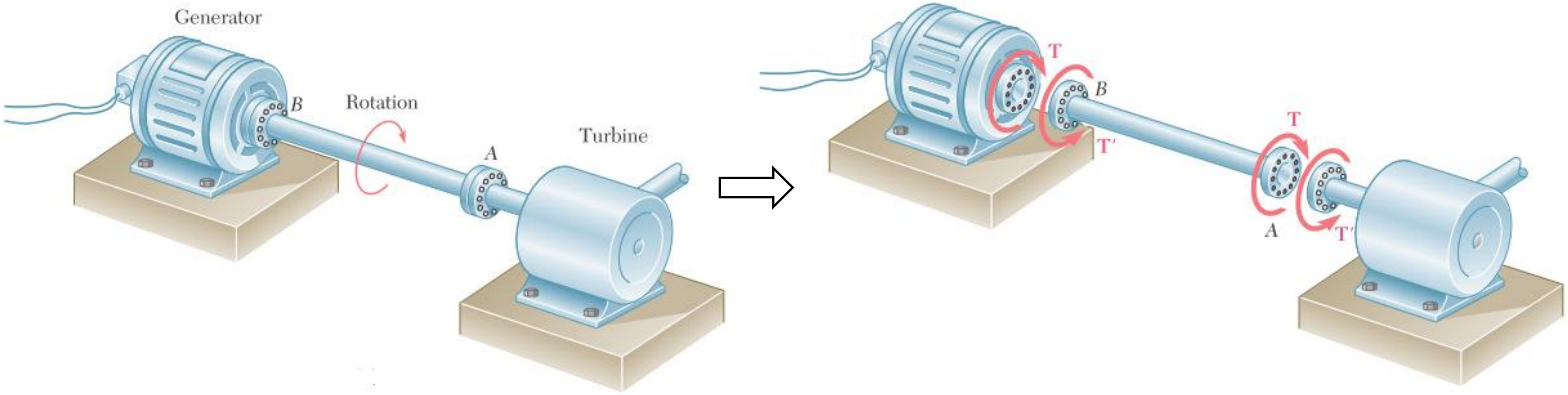
Part I

Objectives

- The concept of torsion in structural members and machine parts
- Shearing stresses and strains in a circular shaft subject to torsion
- Angle of twist in terms of the applied torque, geometry of the shaft, and material
- Torsional deformations to solve indeterminate problems
- Design shafts for power transmission
- Analyze torsion for noncircular members

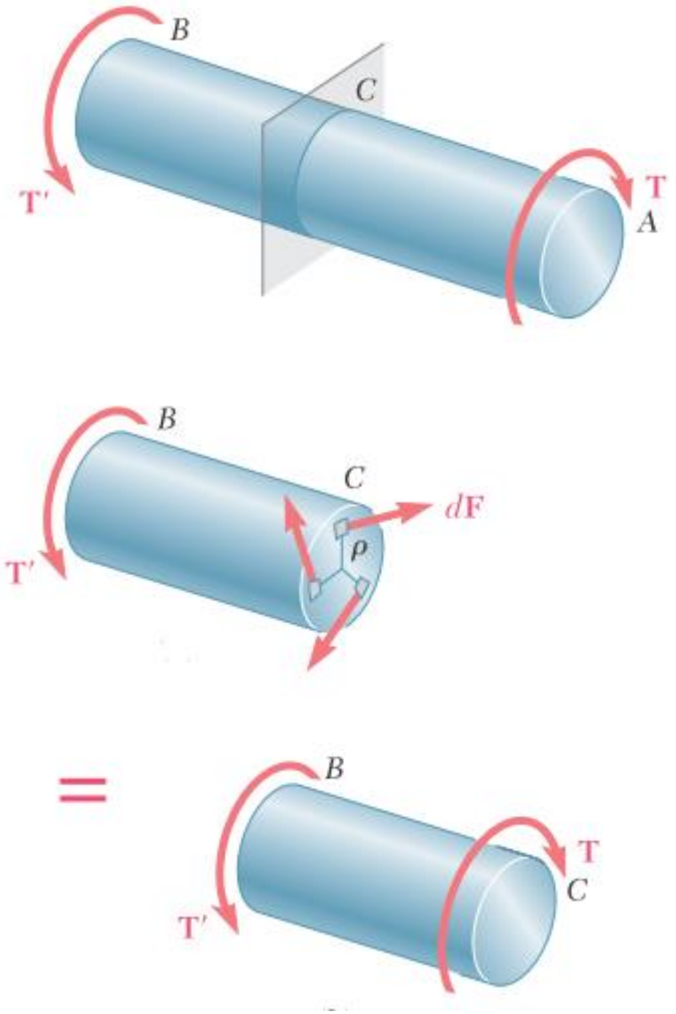


- Members in torsion are encountered in many engineering applications. The most common application is provided by transmission shafts, which are used to transmit power from one point to another. These shafts can be either solid or hollow.



- The system consists of a turbine A and an electric generator B connected by a transmission shaft AB. Breaking the system into its three component parts, the turbine exerts a twisting couple or torque T on the shaft, which then exerts an equal torque on the generator. The generator reacts by exerting the equal and opposite torque T’ on the shaft, and the shaft reacts by exerting the torque T’ on the turbine.

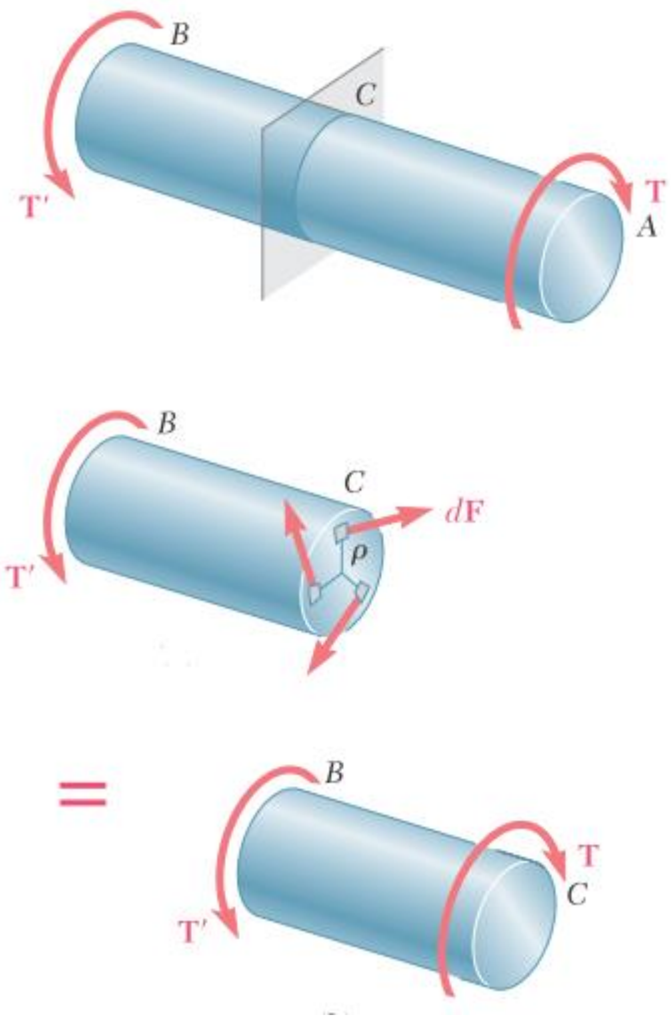
Circular Shafts in Torsion



The Stresses in a Shaft

- The free-body diagram of portion BC of the shaft must include the elementary shearing forces dF, which are perpendicular to the radius of the shaft. These arise from the torque that portion AC exerts on BC as the shaft is twisted.

- The conditions of equilibrium for BC require that the system of these forces be equivalent to an internal torque T, as well as equal and opposite to T'.



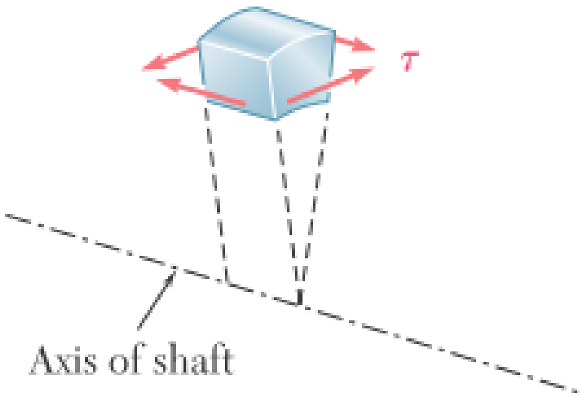
- Denoting the perpendicular distance ρ from the force dF to the axis of the shaft and expressing that the sum of the moments of the shearing forces dF about the axis of the shaft is equal in magnitude to the torque T, write

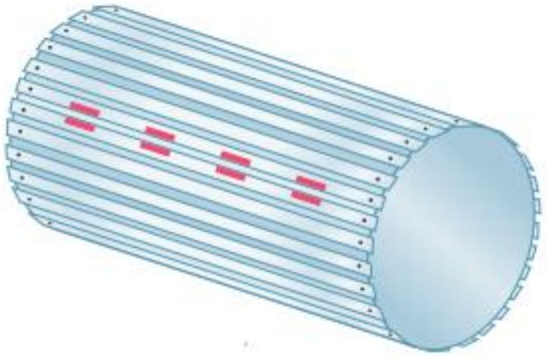
$$\int \rho dF = T$$

- Since $dF = \tau dA$, where τ is the shearing stress on the element of area dA, you also can write

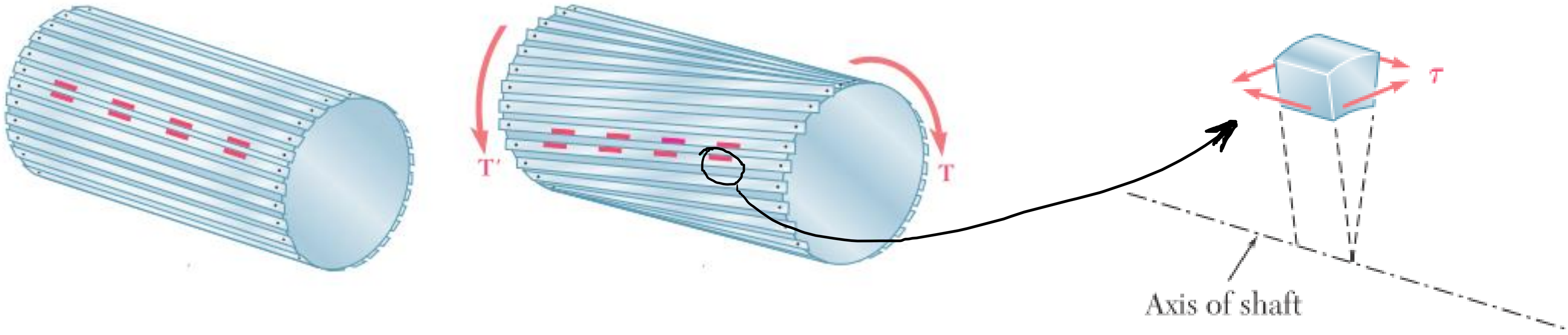
$$\int \rho(\tau dA) = T$$

- While these equations express an important condition that must be satisfied by the shearing stresses in any given cross section of the shaft, they do not tell us how these stresses are distributed in the cross section.
- The actual distribution of stresses under a given load is *statically indeterminate* (i.e., this distribution cannot be determined by the methods of statics).
- However, it was assumed in Chapter 1 that the normal stresses produced by an axial centric load were uniformly distributed, and this assumption was justified in Chapter 2, except in the neighborhood of concentrated loads.
- A *similar assumption* with respect to the *distribution of shearing stresses* in an elastic shaft would be *wrong*. Withhold any judgement until the deformations that are produced in the shaft have been analyzed.



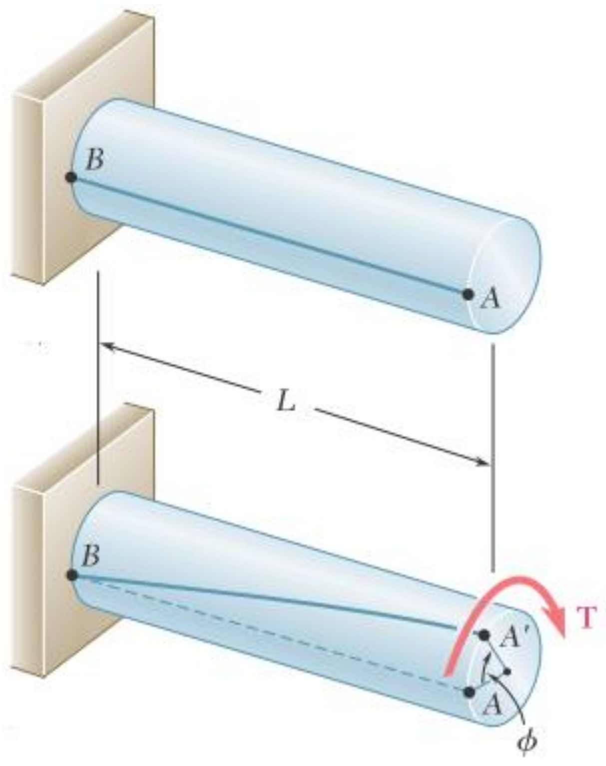


- Shear cannot take place in one plane only. Consider the very small element of shaft shown in figure. The torque applied to the shaft produces shearing stresses τ on the faces perpendicular to the axis of the shaft. However, the conditions of equilibrium require the existence of equal stresses on the faces formed by the two planes containing the axis of the shaft. That such shearing stresses actually occur in torsion can be demonstrated by considering a “shaft” made of separate slats pinned at both ends to disks.



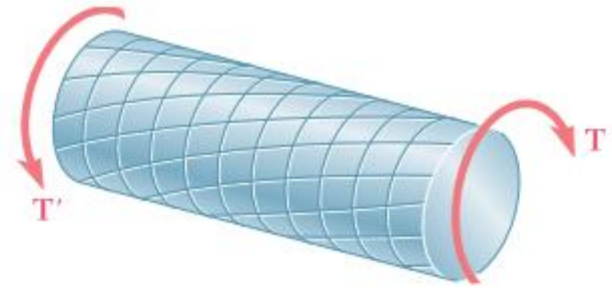
- If markings have been painted on two adjoining slats, it is observed that the slats will slide with respect to each other when equal and opposite torques are applied to the ends of the “shaft”. While sliding will not actually take place in a shaft made of a homogeneous and cohesive material, the tendency for sliding will exist, showing that stresses occur on longitudinal planes as well as on planes perpendicular to the axis of the shaft.

Deformations in a Circular Shaft

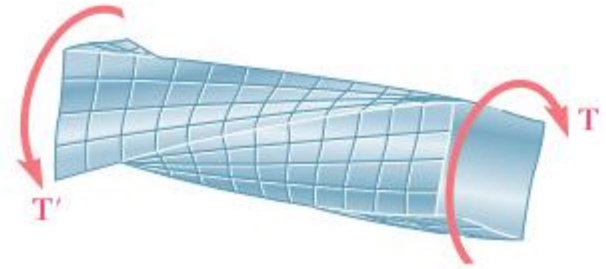


- Consider a circular shaft attached to a fixed support at one end. If a torque T is applied to the other end, the shaft will twist, with its free end rotating through an angle $\boldsymbol{\phi}$ (phi) called the "**angle of twist**".

- Within a certain range of values of T, the angle of twist $\boldsymbol{\phi}$ is proportional to T. Also, ϕ is proportional to the length L of the shaft. In other words, the angle of twist for a shaft of the same material and same cross section, but twice as long, will be twice as large under the same torque T.

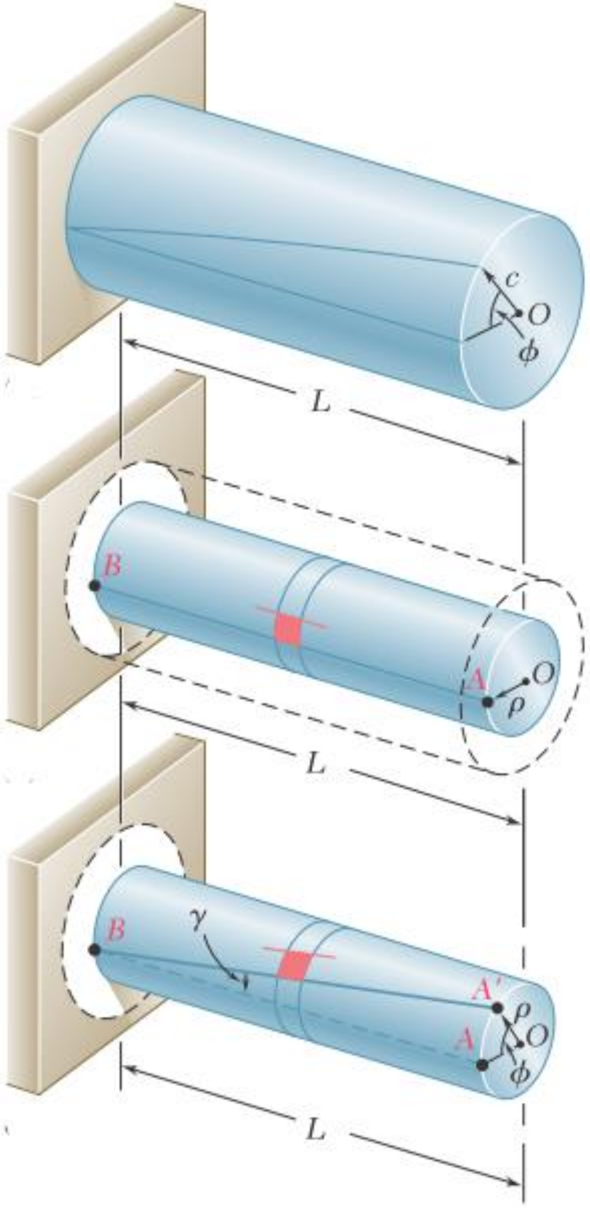


- When a *circular shaft* is subjected to torsion, *every cross section remains plane and undistorted*. In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab.



- This property is characteristic of circular shafts, whether solid or hollow—but not of members with noncircular cross section. For example, when *a bar of square cross section* is subjected to torsion, its various cross sections *warp* and *do not remain plane*.

Shearing Strains



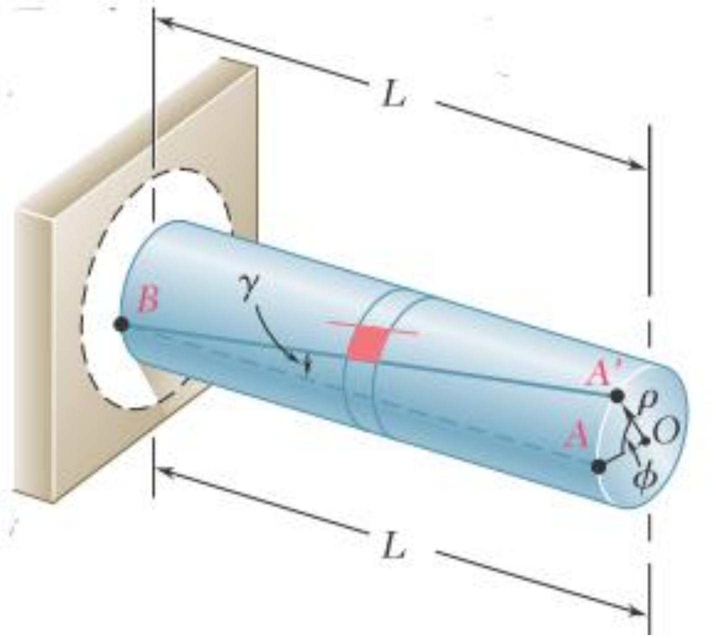
- Detaching from the shaft a cylinder of radius ρ, consider the small square element formed by two adjacent circles and two adjacent straight lines traced on the surface before any load is applied. As the shaft is subjected to a torsional load, the element deforms into a rhombus.

- Here the shearing strain γ in a given element is measured by the change in the angles formed by the sides of that element. Since the circles defining two of the sides remain unchanged, the shearing strain γ must be equal to the angle between lines AB and A'B.

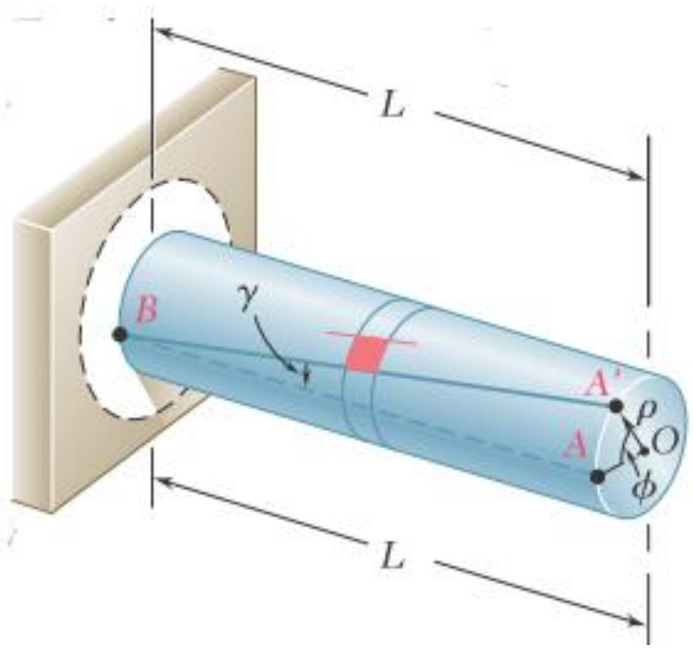
- Figure shows that, for small values of γ, the arc length AA' is expressed as $AA' = L\gamma$. But since $AA' = \rho\phi$, it follows that

$$L\gamma = \rho\phi \; or \; \gamma = \frac{\rho\phi}{L}$$

- where γ and ϕ are in radians.



- This equation shows that the shearing strain γ at a given point of a shaft in torsion is proportional to the angle of twist ϕ. It also shows that γ is proportional to the distance ρ from the axis of the shaft to that point. Thus, the *shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft*.



- The shearing strain is maximum on the surface of the shaft, where $\rho = c$.

$$\gamma_{max} = \frac{c\phi}{L}$$

- Eliminating ϕ from $\gamma = \frac{\rho\phi}{L}$ and $\gamma_{max} = \frac{c\phi}{L}$, the shearing strain γ at a distance ρ from the axis of the shaft is

$$\gamma = \frac{\rho}{c}\gamma_{max}$$

Stresses in the Elastic Range

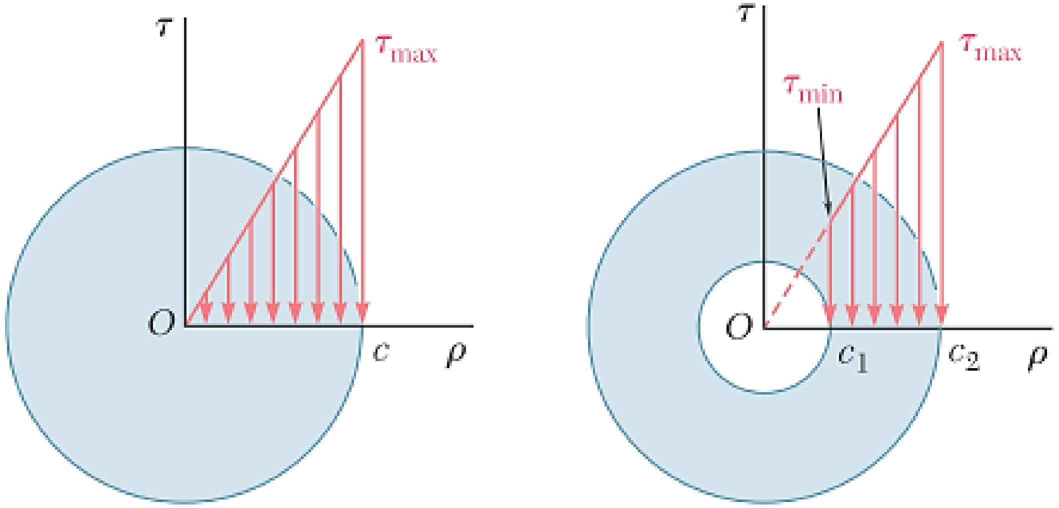
- When the torque T is such that all shearing stresses in the shaft remain below the yield strength τ_y, the stresses in the shaft will remain below both the proportional limit and the elastic limit. Thus, Hooke's law will apply, and there will be no permanent deformation. Recalling Hooke's law for shearing stress and strain

$$\tau = G\gamma$$

- where G is the modulus of rigidity or shear modulus of the material. Multiplying both members of $\gamma = \frac{\rho}{c}\gamma_{max}$ by G

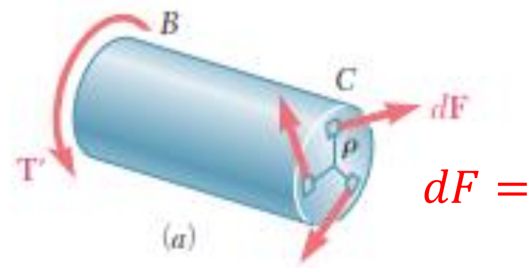
$$G\gamma = G\frac{\rho}{c}\gamma_{max} \qquad or \qquad \boxed{\tau = \frac{\rho}{c}\tau_{max}}$$

- This equation shows that, as long as the yield strength (or proportional limit) is not exceeded in any part of a circular shaft, *the shearing stress in the shaft varies linearly with the distance ρ from the axis of the shaft*.



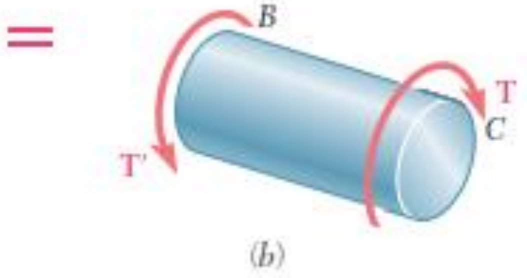
$$\tau_{min} = \frac{c_1}{c_2}\tau_{max}$$

- The sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:



$dF = \tau dA$

$$\int \rho(\tau dA) = T$$



- Substituting shearing stress formula $\tau = \frac{\rho}{c}\tau_{max}$

$$T = \int \rho\tau dA = \frac{\tau_{max}}{c}\int \rho^2 dA$$

- The integral in the last part represents the polar moment of inertia J of the cross section with respect to its center O. Therefore,

$$T = \frac{\tau_{max}}{c}J \qquad or \qquad \boxed{\tau_{max} = \frac{Tc}{J}}$$

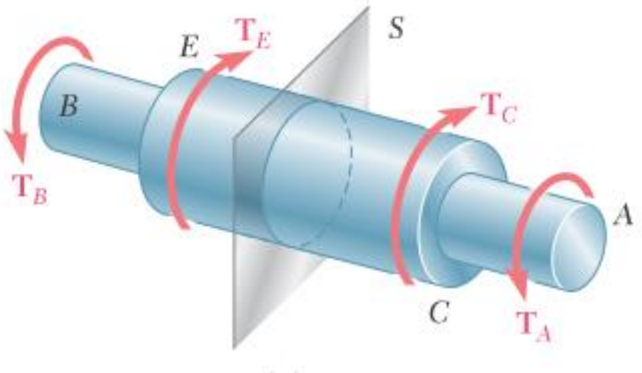
Max. shearing stress on circular shaft

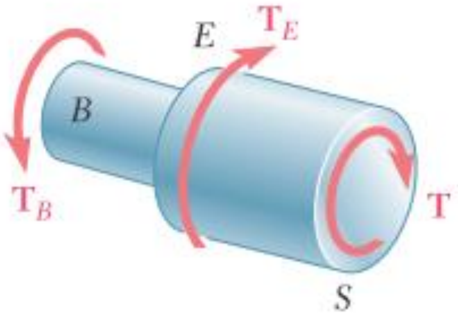
- Substituting for τ_{max} from $\tau_{max} = \frac{Tc}{J}$ into $\tau = \frac{\rho}{c}\tau_{max}$, the shearing stress at any distance ρ from the axis of the shaft is

$$\tau = \frac{T\rho}{J}$$

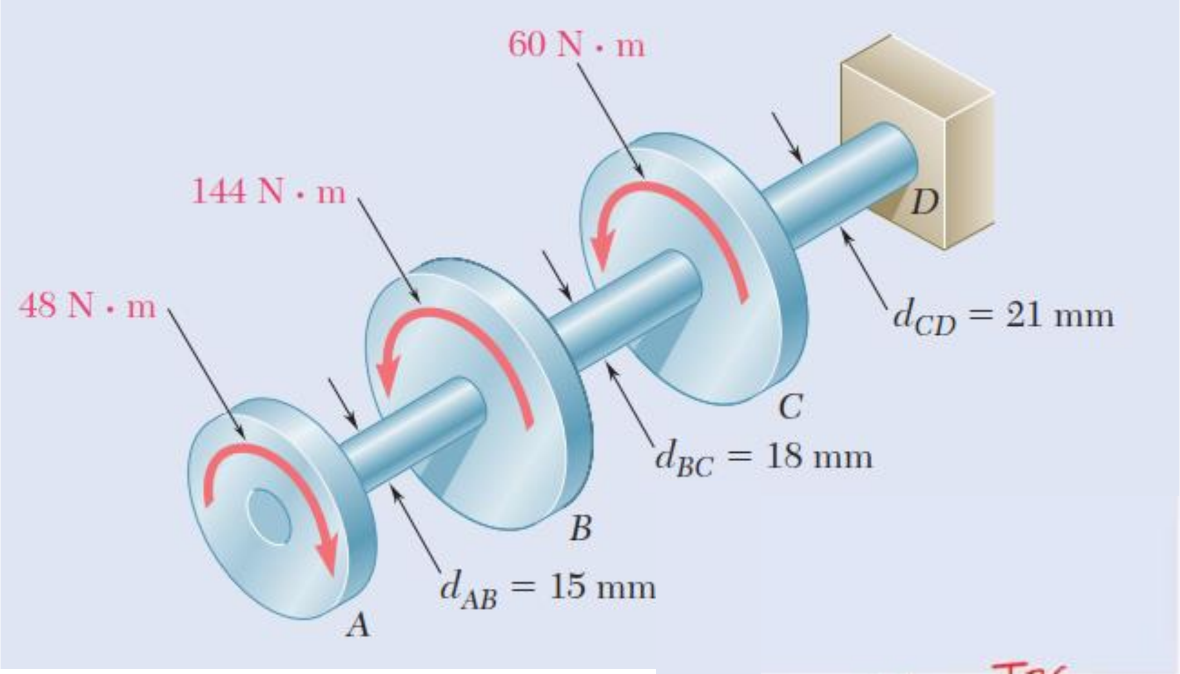
Shearing stress at any distance on circular shaft

- The torsion formulas were derived for a shaft of uniform circular cross section subjected to torques at its ends.

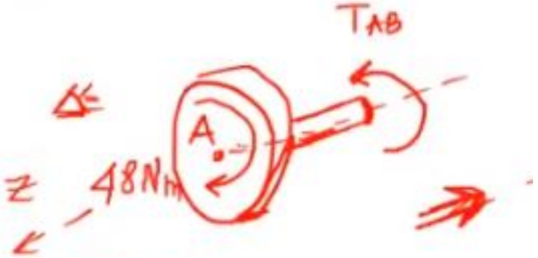




Sample Problem 3.11



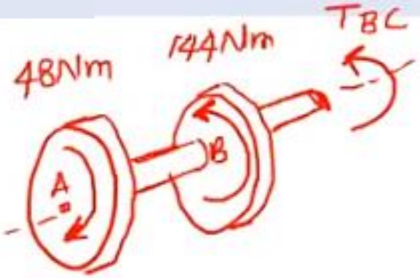
Knowing that each of the shafts AB, BC, and CD consist of a solid circular rod, determine (a) the shaft in which the maximum shearing stress occurs, (b) the magnitude of that stress.



$\Sigma T = 0$

$= -48 + T_{AB}$

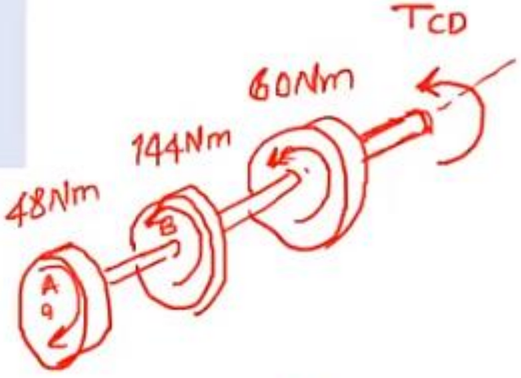
$\boxed{T_{AB} = 48 \text{ Nm CCW}}$



$\Sigma T = 0$

$= -48 + 144 + T_{BC}$

$\boxed{T_{BC} = -96 \text{ Nm}}$ (CW)



$\Sigma T = 0$

$= -48 + 144 + 60 + T_{CD}$

$\boxed{T_{CD} = -156 \text{ Nm}}$ (CW)

Sample Problem 3.11

(SHAFT AB) $$(\tau_{AB})_{max} = \frac{(48\ Nm)(0.0075\ m)}{\frac{\pi\ (0.015)^4}{32}\ m^4} = \underline{\underline{72.4\ MPa}} \quad (72.4 \times 10^6\ Pa) \rightarrow \frac{N}{m^2}$$

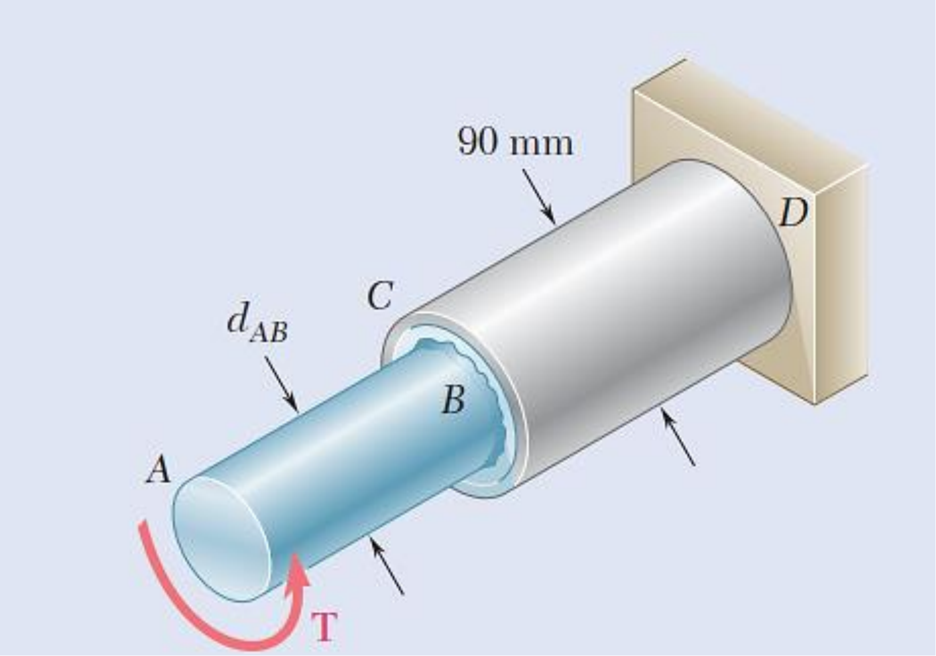
(BC) $$(\tau_{BC})_{MAX} = \frac{(-96\ Nm)(0.009\ m)}{\frac{\pi\ (0.018)^4}{32}\ m^4} = -83.8\ MPa$$

(CD) $$(\tau_{CD})_{MAX} = \frac{(-156\ Nm)(0.0105\ m)}{\frac{\pi\ (0.021)^4}{32}\ m^4} = \underline{-85.8\ MPa}$$

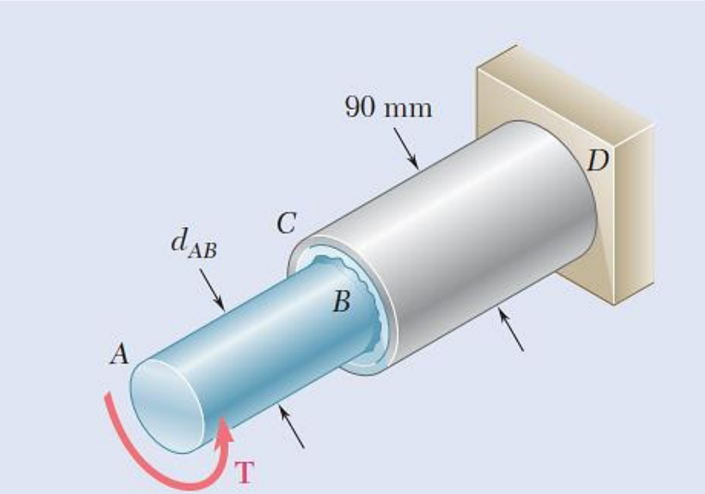
(a) τ_{Max} will be observed in shaft CD.

(b) $\tau_{max} = -85.8\ MPa$

Sample Problem 3.19



The solid rod *AB* has a diameter d_{AB} = 60 mm and is made of a steel for which the allowable shearing stress is 85 MPa. The pipe *CD*, which has an outer diameter of 90 mm and a wall thickness of 6 mm, is made of an aluminum for which the allowable shearing stress is 54 MPa. Determine the largest torque **T** that can be applied at *A*.



$\tau_{all.} = \dfrac{T_{all.} c}{J}$

$J_{circular} = \dfrac{\pi c^4}{2}$ (SOLID)

$J_{circular} = \dfrac{\pi}{2}(C_{outer}^4 - C_{inner}^4)$
(HOLLOW)

Rod AB $\quad \tau_{all} = 85\ \text{MPa} \quad c = \dfrac{d}{2} = 0.03\ \text{m}$

Allowable torque (T_{all})

$$T_{all} = \frac{\tau_{all} \cdot J}{c} = \frac{(85 \times 10^6)\left(\frac{\pi \cdot 0.03^4}{2}\right)}{0.03}$$

$\Rightarrow T_{all} = 3.605 \times 10^3\ \text{N.m}$

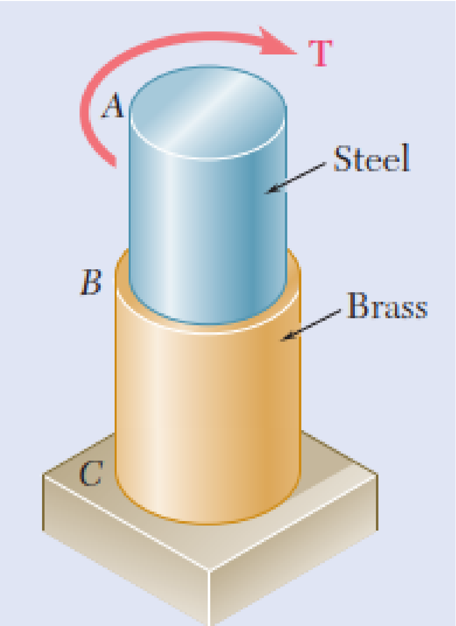
Pipe CD
- $\tau_{all} = 54 \times 10^6\ \text{Pa}$
- $c_{outer} = 0.045\ \text{m}$
- $c_{inner} = 0.039\ \text{m}$

$$T_{all} = \frac{(54 \times 10^6)\left(\frac{\pi(0.045^4 - 0.039^4)}{2}\right)}{0.045}$$

$\Rightarrow T_{all} = 3.365 \times 10^3\ \text{N.m}$

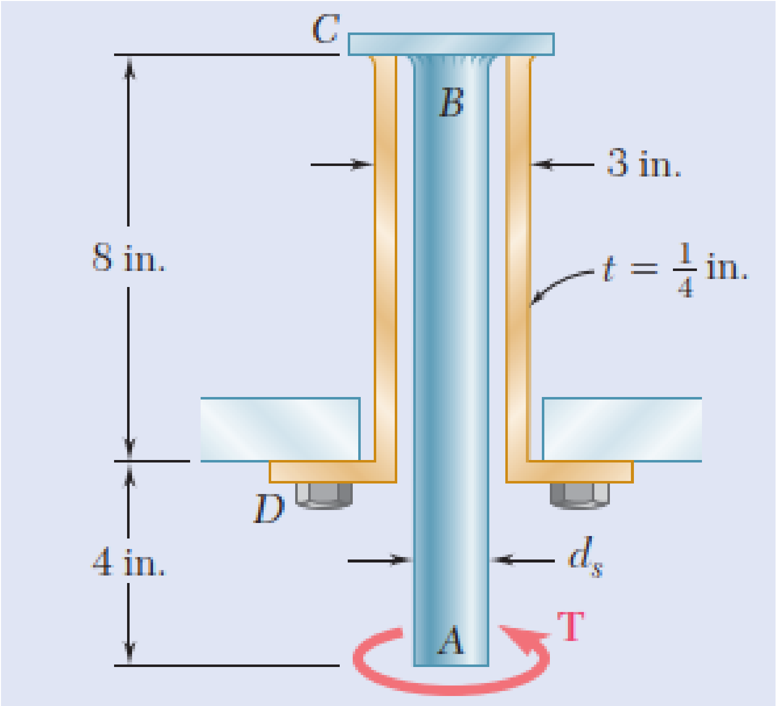
* SMALLER TORQUE IS THE ALLOWABLE TORQUE
$T_{all} = 3.365 \times 10^3\ \text{m}$ //

Sample Problem 3.15



The allowable shearing stress is 15 ksi in the 1.5-in.-diameter steel rod *AB* and 8 ksi in the 1.8-in.-diameter brass rod *BC*. Neglecting the effect of stress concentrations, determine the largest torque **T** that can be applied at *A*.

Sample Problem 3.7



The solid spindle *AB* is made of a steel with an allowable shearing stress of 12 ksi, and sleeve *CD* is made of a brass with an allowable shearing stress of 7 ksi. Determine (*a*) the largest torque **T** that can be applied at *A* if the allowable shearing stress is not to be exceeded in sleeve *CD*, (*b*) the corresponding required value of the diameter d_s of spindle *AB*.