ME 209
Numerical Methods

3. Finding Roots of the Equations:
Bracketing Methods

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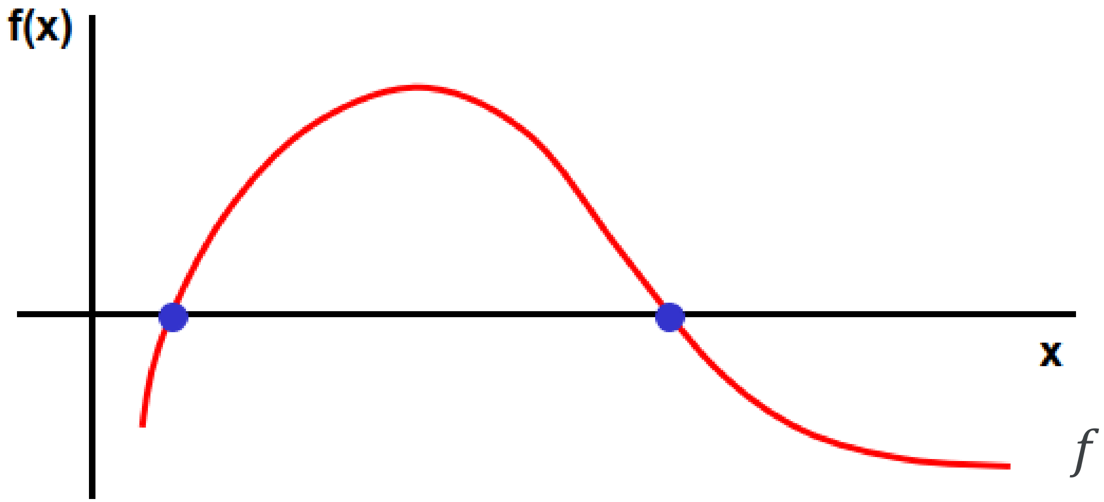
2024

3.1. INTRODUCTION



$f(x) = x \qquad x = ?$

- Finding the root of an equation $f(x)$ or solving that equation means determining the x values that make the equation zero. For this reason, the roots of equations are sometimes called the zeros of these equations.
- We know how to find the roots of 2nd degree polinomial function $f(x) = ax^2 + bx + c = 0$

by using the equation $x = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

3.2. FINDING ROOTS OF EQUATIONS

- However, there are many complex equations whose roots we cannot easily find analytically. In such cases, numerical methods can be used as powerful alternatives in finding the roots of equations.
- Numerical methods commonly used in finding the roots of equations can be examined in two groups. These are: ***Bracketing methods*** and ***open methods***.

Bracketing methods:

- Bisection Method
- False-Position Method

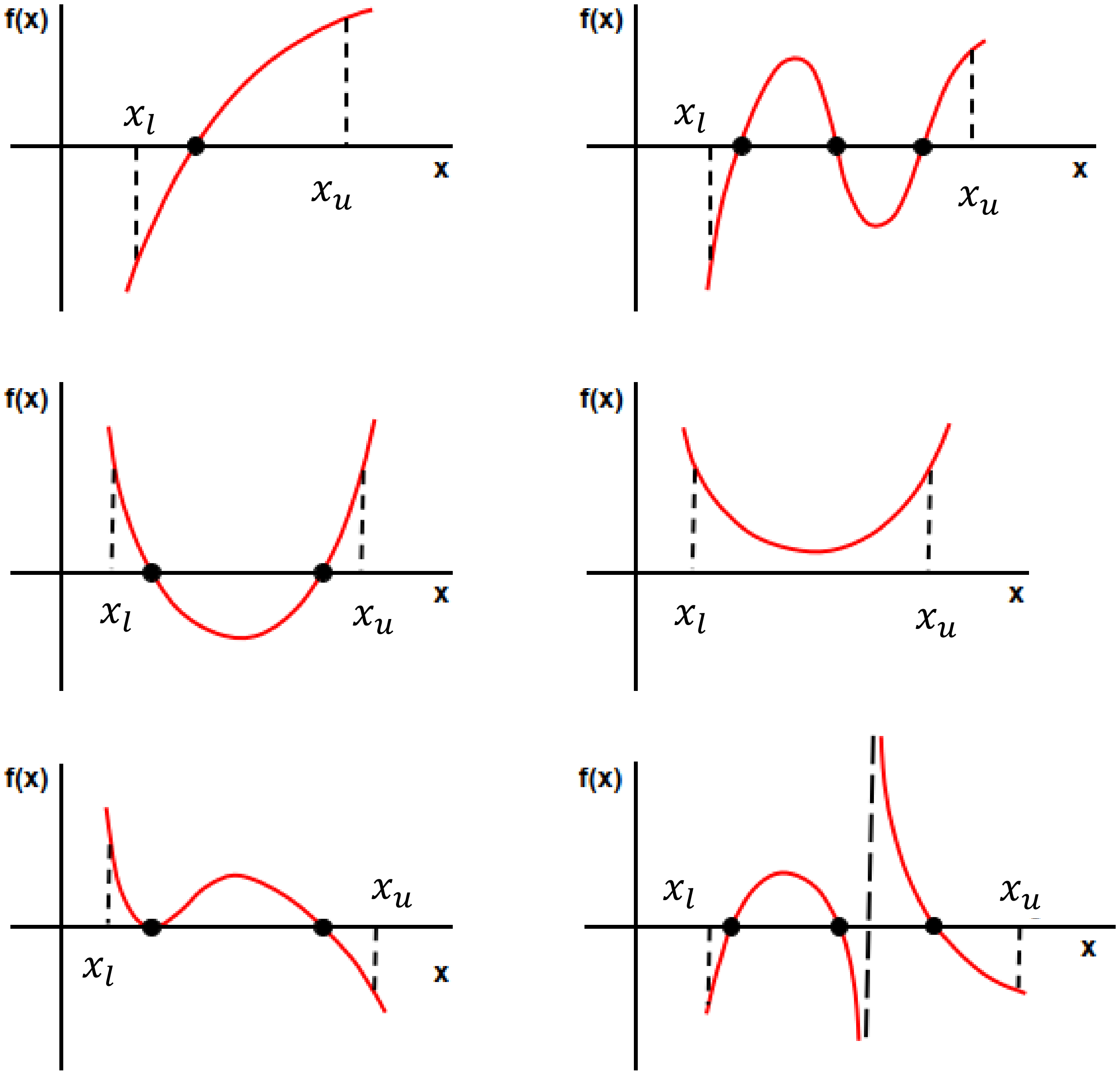
Open methods:

- One-point Iteration Method
- Newton-Raphson Method
- Secant Method

3.3. Bracketing Methods

- These techniques deal with techniques that take use of the fact that a function frequently changes sign close to a root.
- And these techniques are called bracketing methods because ***two initial guesses for the root are required***.
- As the name implies, these guesses must “bracket,” or be on either side of, the root.
- The particular methods described herein employ different strategies to systematically reduce the width of the bracket and, hence, focus on the correct answer.

3.3.1. General Idea of Bracketing Methods



RULE 1: IF $f(x_l) * f(x_u) < 0$ THEN

there are odd number of roots.

RULE 2: IF $f(x_l) * f(x_u) > 0$ THEN

there are: (i) even number of roots,

(ii) no roots.

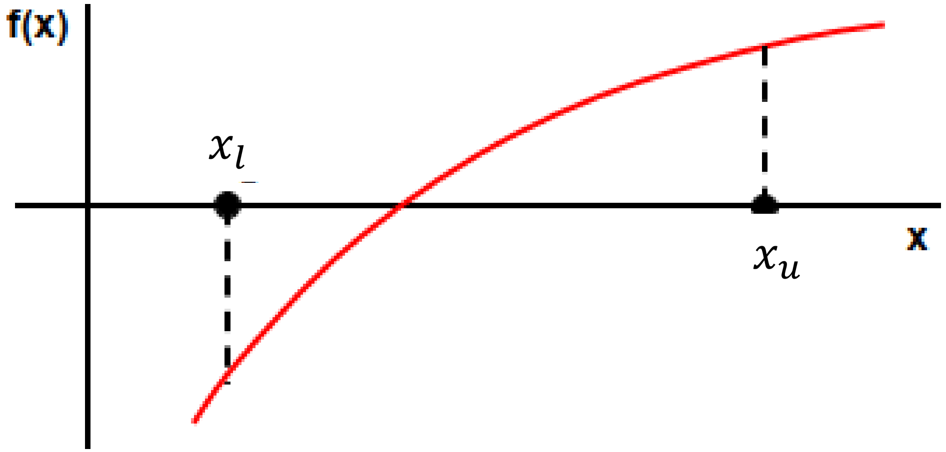
VIOLATIONS:

(i) multiple roots,

(ii) discontinuities.

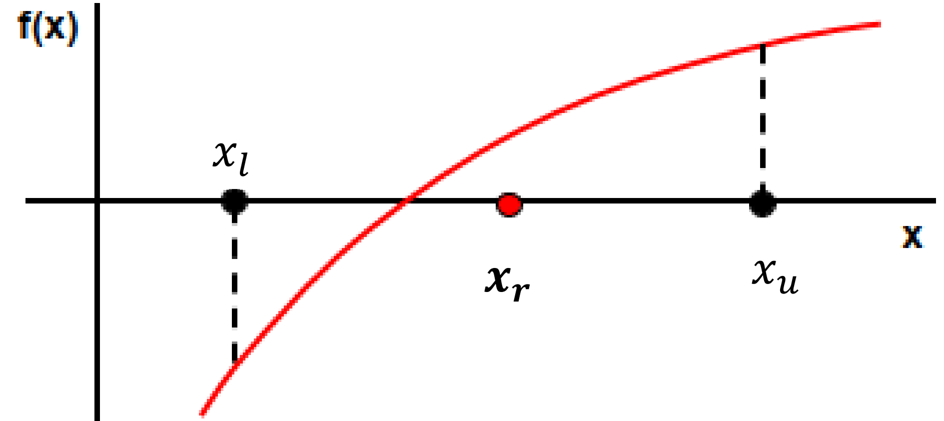
3.3.2. THE BISECTION METHOD

The ***bisection method*** is one type of incremental search method in which the interval is always divided in half. If a function changes sign over an interval, the function value at the midpoint is evaluated.

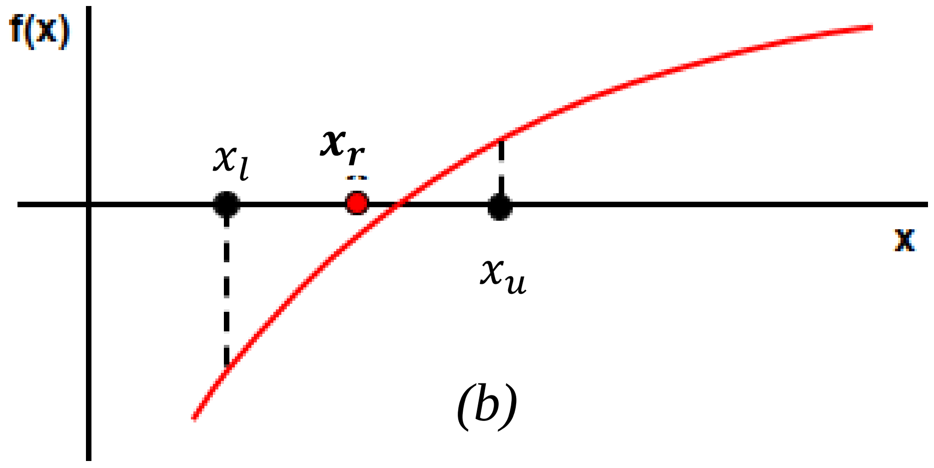


- **Step 1:** Choose two initial estimations, x_{LOWER} (x_l) and x_{UPPER} (x_u).
- They should bracket the root, i.e.

$$f(x_l) * f(x_u) < 0$$



(a)



(b)

Step 2: Estimate the root as a midpoint of this interval

$$x_r = \frac{(x_l + x_u)}{2}$$

Step 3: Determine the interval which contains the root,

(a) IF $f(x_l) * f(x_r) < 0$ THEN the root is between x_L and x_r, Therefore, set $x_u = x_r$ and RETURN to step 2.

(b) IF $f(x_l) * f(x_r) > 0$ THEN the root is between x_r and x_U. Therefore, set $x_l = x_r$ and RETURN to step 2.

(c) IF $f(x_l) * f(x_r) = 0$, the root is equals x_r; TERMINATE the computation

Termination Criteria and Error Estimates

We need a termination criteria to end the iteration. Here, an approximate percent relative error ε_a can be calculated as:

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%$$

When the approximate percent relative error ε_a falls below the specified percent relative error ε_s or tolerance, we could terminate the bisection method.

Example 3.1: Find the square root of 11. (Tolerance value: $|\varepsilon_s| = 0.5\%$)

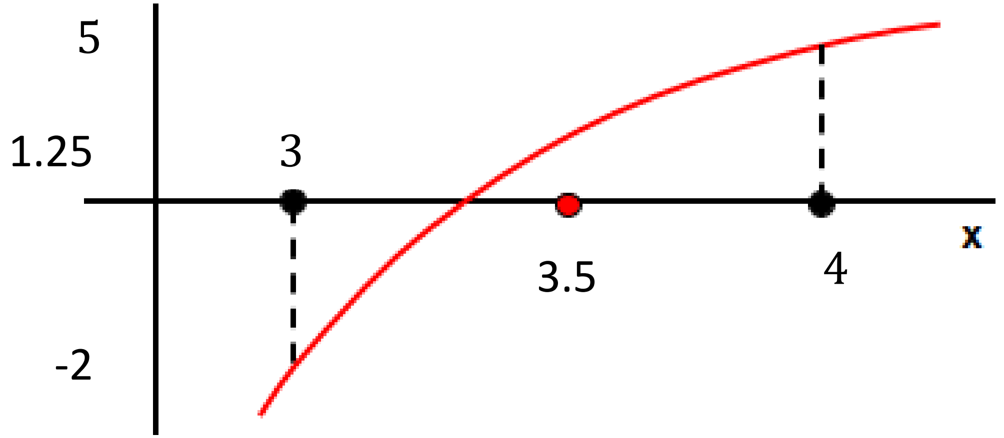
$$x^2 = 11 \ \rightarrow f(x) = x^2 - 11 \quad (exact\ solution\ is\ 3.31662479)$$

Choose initial estimates: $3^2 = 9 < 11$ and $4^2 = 16 > 11 \ \rightarrow \boldsymbol{x_l} = \mathbf{3}$ and $\boldsymbol{x_u} = \mathbf{4}$

1st iteration: $x_r = \frac{x_l + x_u}{2} = \frac{3+4}{2} \rightarrow x_r = 3.5$ and $f(3.5) = 3.5^2 - 11 = 1.25$

$f(x_l) * f(x_r) = -2 * 1.25 \rightarrow -2.5 < 0$ (root is between x_l and x_r)

SET $x_u = x_r$=3.5 and CONTINUE to iterate.



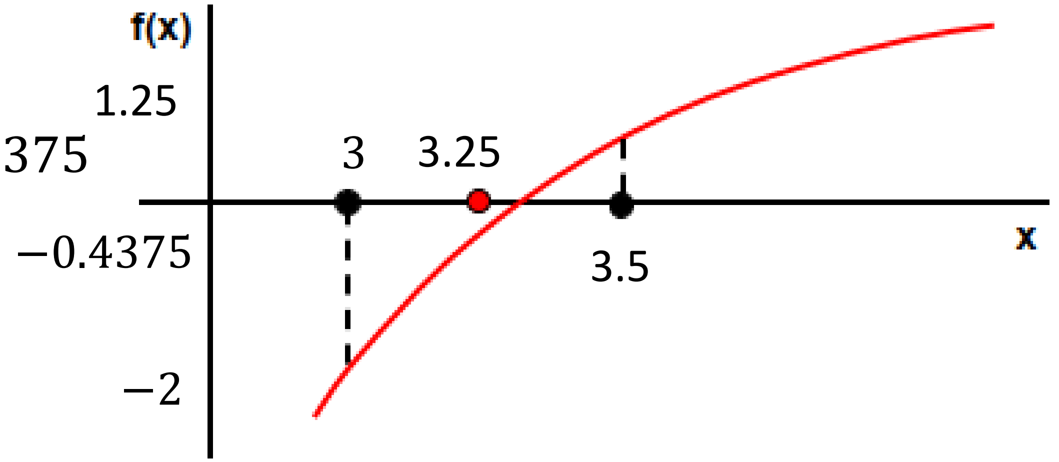
2nd iteration: $\boldsymbol{x_l = 3, x_u = 3.5}$

$$x_r = \frac{3+3.5}{2} \rightarrow x_r = 3.25 \text{ and } f(3.25) = 3.25^2 - 11 = -0.4375$$

$$f(x_l) * f(x_r) = -2 * -0.4375 = 0.875 > 0$$

(root is between x_r and x_u)

SET $x_l = x_r = 3.25$ and CONTINUE to iterate.



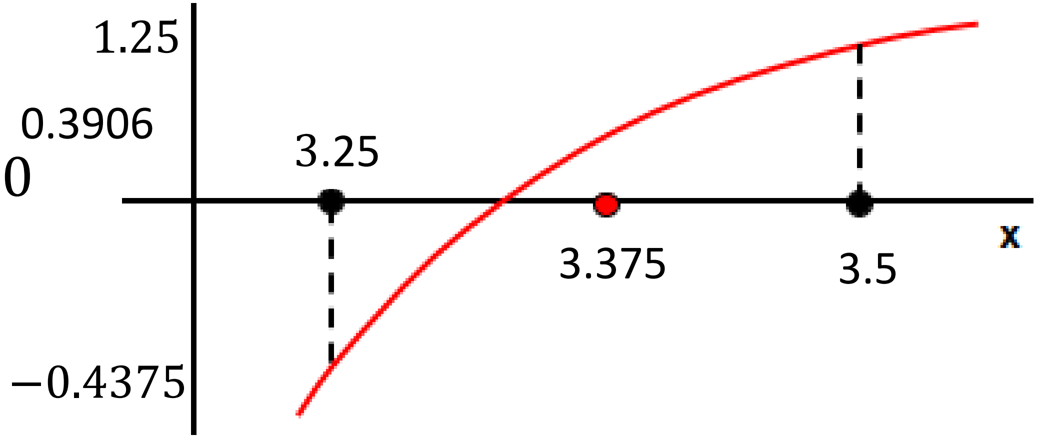
3rd iteration: $\boldsymbol{x_l = 3.25, x_u = 3.5}$

$$x_r = \frac{3.25+3.5}{2} \rightarrow x_r = 3.375 \text{ and } f(3.375) = 0.390625$$

$$f(x_l) * f(x_r) = -0.4375 * 0.390625 = -0.17089 < 0$$

(root is between x_l and x_r)

SET $x_u = x_r = 3.375$ and CONTINUE to iterate.

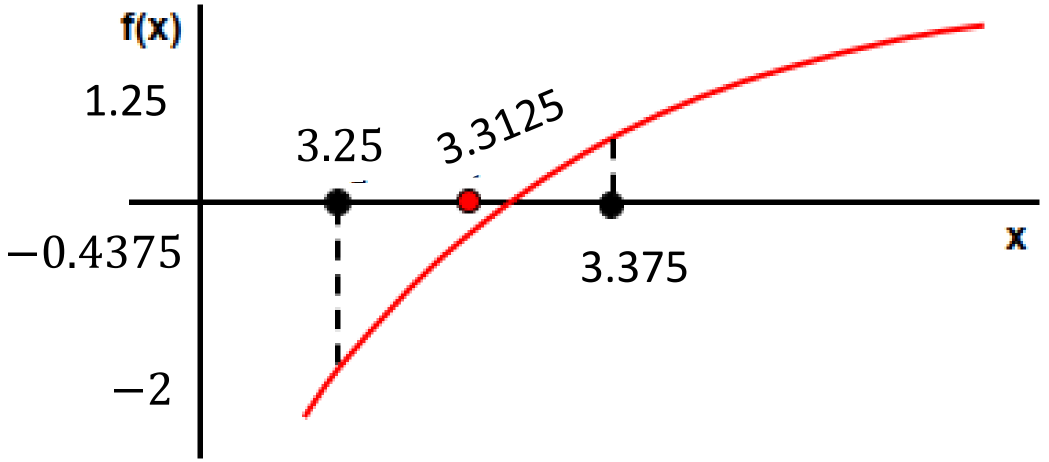


4th iteration: $\boldsymbol{x_l = 3.25, x_u = 3.375}$

$x_r = \frac{3.25+3.375}{2} \rightarrow x_r = 3.3125$
and $f(3.3125) = 3.3125^2 - 11 = -0.027343$

$-0.4375 * -0.027343 = 0.011963 > 0$
(root is between x_r and x_u)

SET $x_l = x_r = 3.3125$ and CONTINUE to iterate.

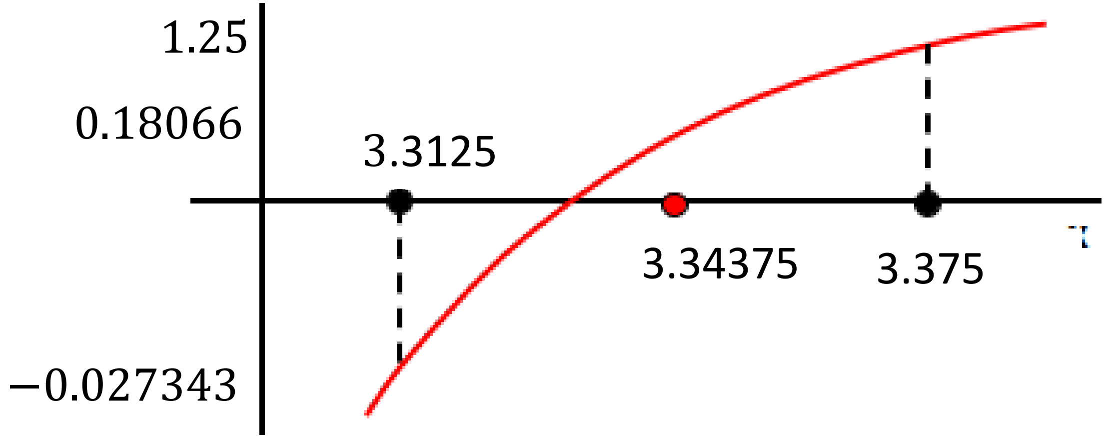


5th iteration: $\boldsymbol{x_l = 3.3125, x_u = 3.375}$

$x_r = \frac{3.3125+3.375}{2} \rightarrow x_r = 3.34375$ and
$f(3.34375) = 0.18066$

$-0.027343 * 0.18066 = -0.01068 < 0$
(root is between x_l and x_r)

SET $x_u = x_r = 3.34375$ and CONTINUE to iterate.



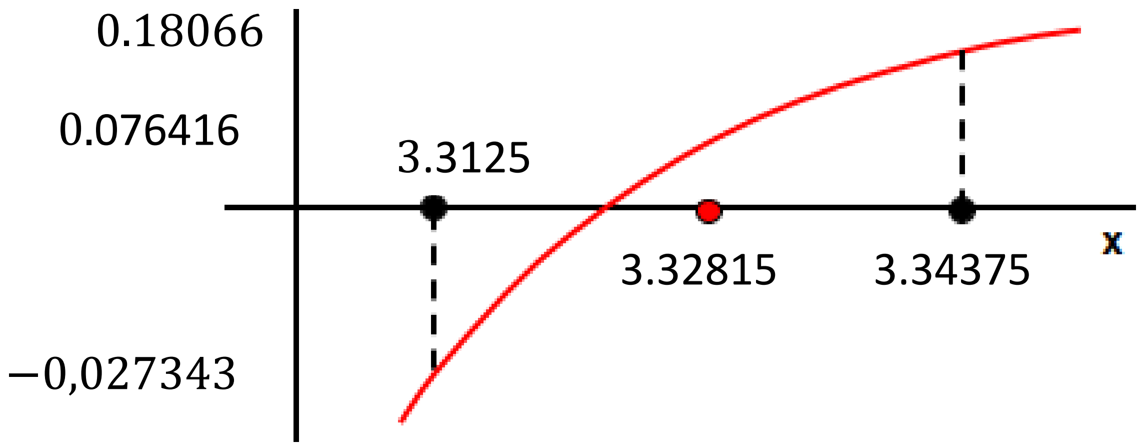
6th iteration: $\boldsymbol{x_l = 3.3125, x_u = 3.34375}$

$$x_r = \frac{3.3125+3.34375}{2} \rightarrow x_r = 3.32815 \text{ and}$$

$$f(3.32815) = 0.076416$$

$$-0.027343 * 0.076416 = -0.002089 < 0$$

(root is between x_l and x_r)



Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_r)$	$f(x_l)f(x_r)$	$\|\varepsilon_a\|$ (%)
1	3	3.5	4	-2	1.25	-2.5	-
2	3	3.25	3.5	-2	-0.4375	0.875	7.69
3	3.25	3.375	3.5	-0.4375	0.390625	-0.17089	3.70
4	3.25	3.3125	3.375	-0.4375	-0.027343	0.011963	1.88
5	3.3125	3.34375	3.375	-0.027343	0.18066	-0.004942	0.93
6	3.3125	$\boxed{3.32815}$	3.34375	-0.027343	0.076582	-0.0020939	$\boxed{0.468}$

$$\sqrt{11} \approx 3.32815$$

$$|\varepsilon_a| < |\varepsilon_s|$$

Example 3.2 Calculate root of following polynolmial

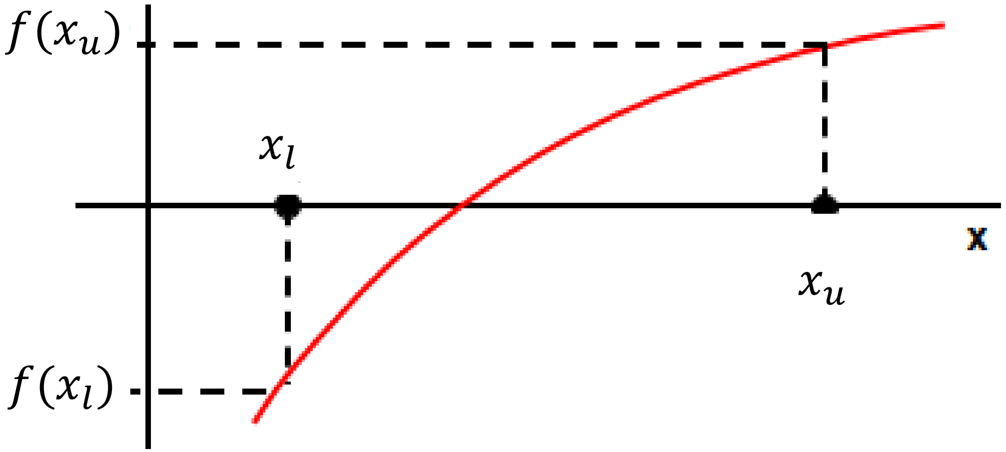
$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

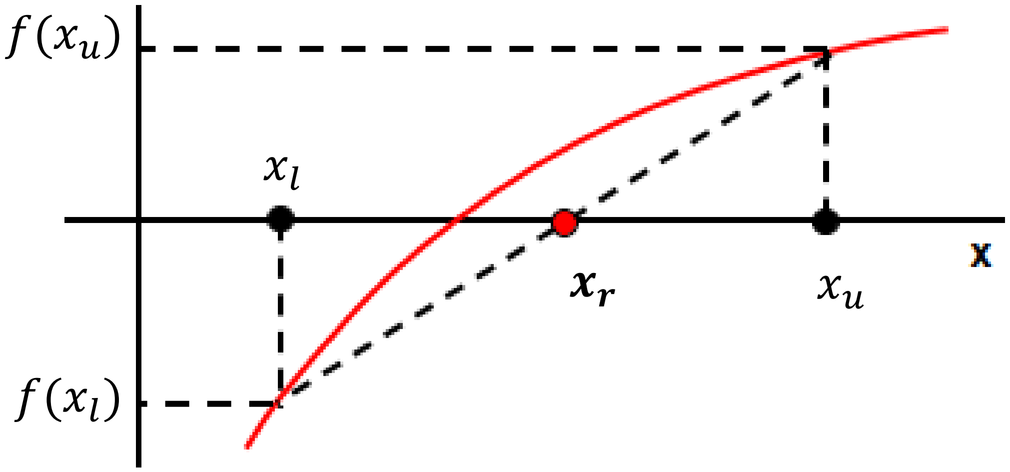
Finding the roots within the interval 3.75 $\leq x \leq$ 5.00 to a relative accuracy as an absolute value between successive iterations of 0.01.

Check: f(x=3.75) = -6.82
f(x=5.00) = 42
fl x fu = -286.44 < 0

Iteration i	x_l	x_r	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	Subinterval Containing Root $f(x_l)f(x_r)$	$f(x_r)f(x_u)$	Error ϵ
1	3.750	4.375	5.000	–6.830	12.850	42.000	–	+	–
2	3.750	4.062	4.375	–6.830	1.903	12.850	–	+	0.313
3	3.750	3.906	4.062	–6.830	–2.724	1.903	+	–	0.156
4	3.906	3.984	4.062	–2.724	–0.477	1.903	+	–	0.078
5	3.984	4.023	4.062	–0.477	0.696	1.903	–	+	0.039
6	3.984	4.004	4.023	–0.477	0.120	0.696	–	+	0.019
7	3.984	3.994	4.004	–0.477	–0.180	0.120	+	–	0.010

3.3.3. FALSE-POSITION METHOD





- **Step 1:** Choose two initial estimations, x_{LOWER} (x_l) and x_{UPPER} (x_u).
- They should bracket the root, i.e.

$$f(x_l) * f(x_u) < 0$$

- **Step 2:** Using similar triangles, the intersection of the straight line with the x axis can be estimated as

$$x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

or

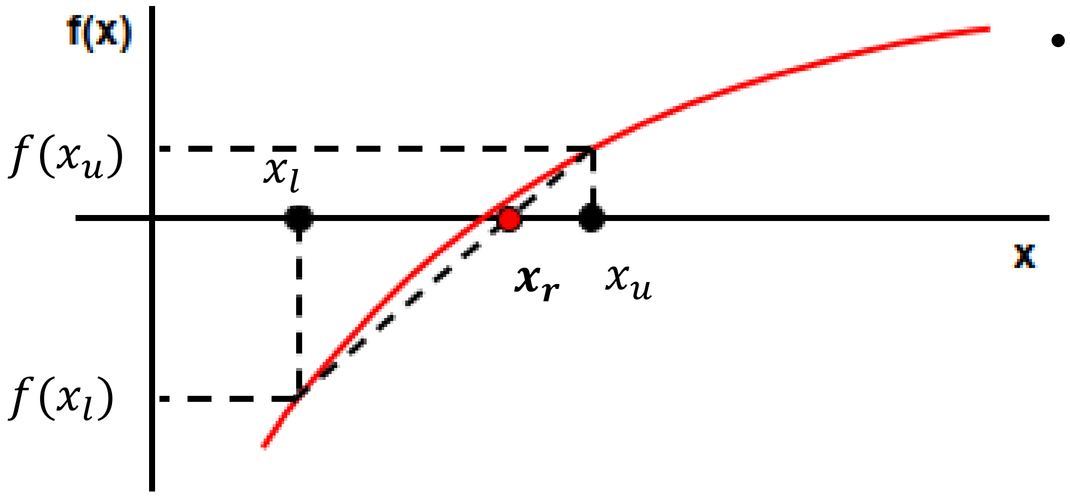
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

- Determine the interval which contains the root:

  IF $f(x_l) * f(x_r) < 0$ root is between x_l and x_r

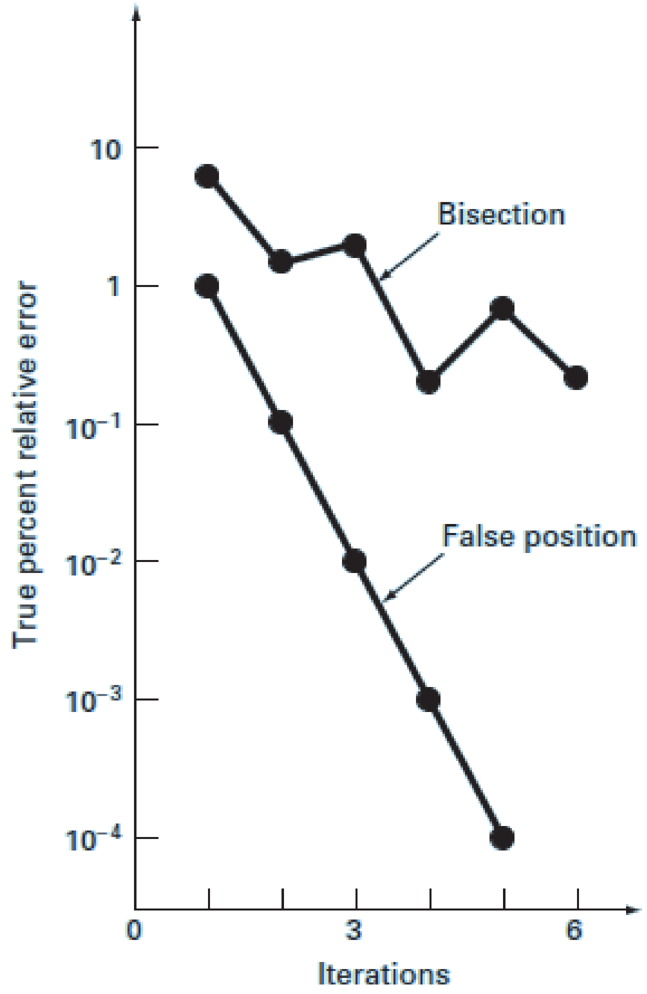
  ELSE root is between x_r and x_u

3.3.3. FALSE-POSITION METHOD



- **Step 3:** Estimate a new root in this interval,

  Stop when the specified tolerance is reached.



- False-position method always converges to the true root,
- $f(x_l) * f(x_u) < 0$ is true if the interval has odd number of roots, not necessarily one root.
- The *false-position method* generally converges faster than the *bisection method*.

Example 3.2: Find the square root of 11 by using false-position method. (Tolerance value: $|\varepsilon_s| = 0.5\%$)

$$x^2 = 11 \;\rightarrow f(x) = x^2 - 11 \quad (exact\ solution\ is\ 3.31662479)$$

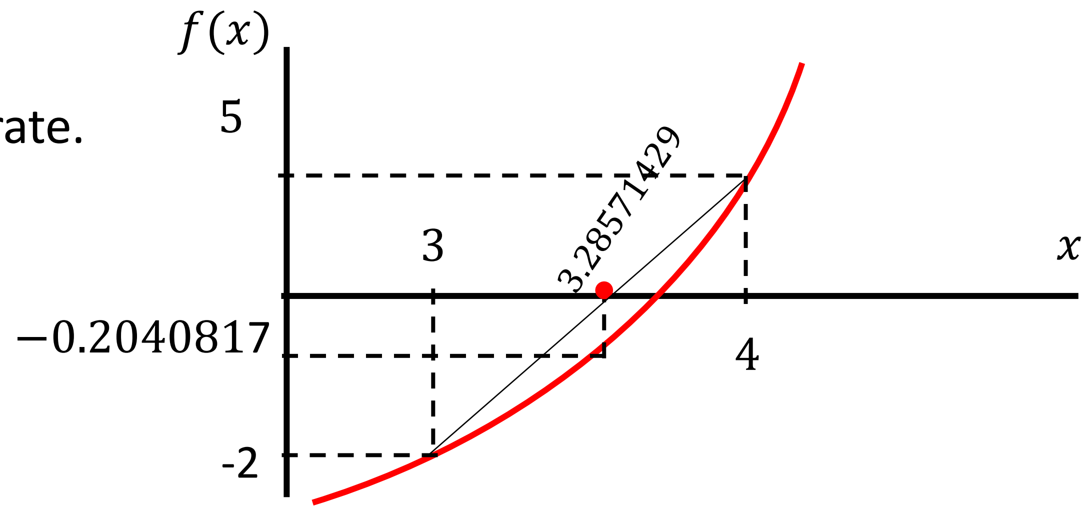
Choose initial estimates: $3^2 = 9 < 11$ and $4^2 = 16 > 11 \;\rightarrow \boldsymbol{x_l} = \mathbf{3}$ and $\boldsymbol{x_u} = \mathbf{4}$

1st iteration: $x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.28571429$ and

$f(3.28571429) = 3.28571429^2 - 11 = -0.2040817$

$f(x_l) * f(x_r) = -2 * -0.2040817 \rightarrow 0.408163 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$= 3.28571429 and CONTINUE to iterate.



2nd iteration: $\boldsymbol{x_l = 3.28571429, x_u = 4}$

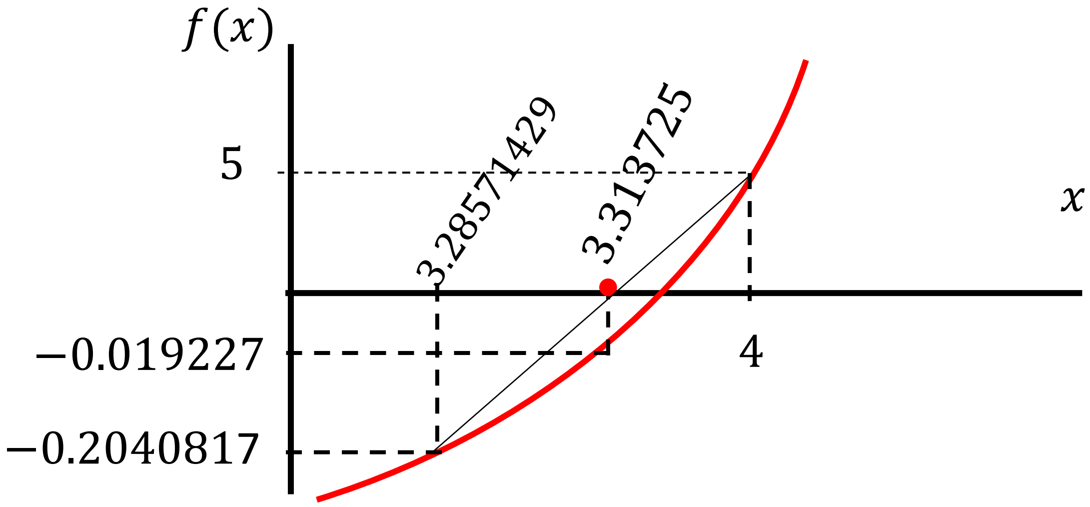
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.313725$ and

$f(3.313725) = 3.313725^2 - 11 = -0.019227$

$f(x_l) * f(x_r) = -0.2040817 * -0.019227$

$\rightarrow 0.00392 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$= 3.313725 and CONTINUE to iterate.



3rd iteration: $\boldsymbol{x_l = 3.313725, x_u = 4}$

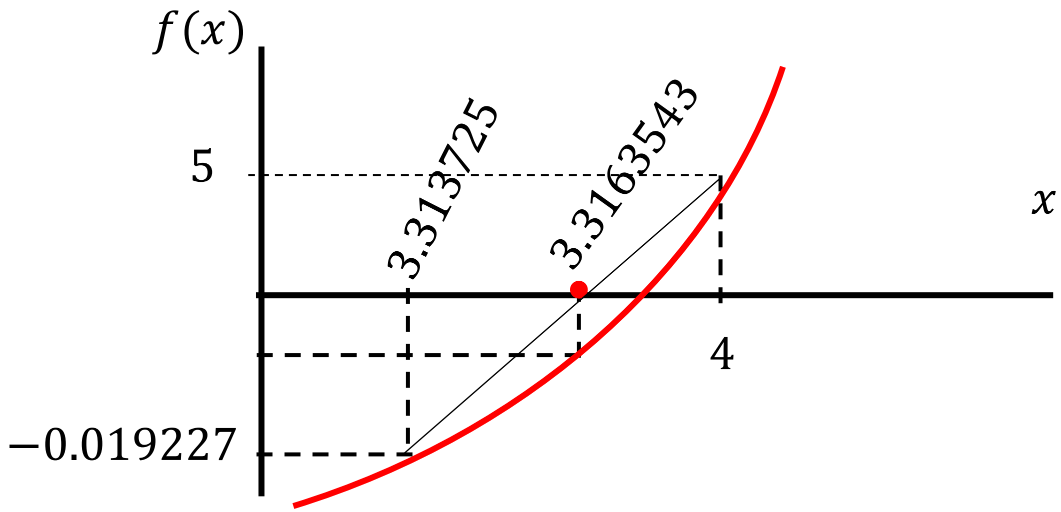
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.3163543$ and

$f(3.3163543) = 3.3163543^2 - 11 = -0.00179415$

$f(x_l) * f(x_r) = -0.019227 * -0.00179415$

$\rightarrow 0.0000344961 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$=3.3163543 and CONTINUE to iterate.



4th iteration: $x_l = 3.3163543, x_u = 4$

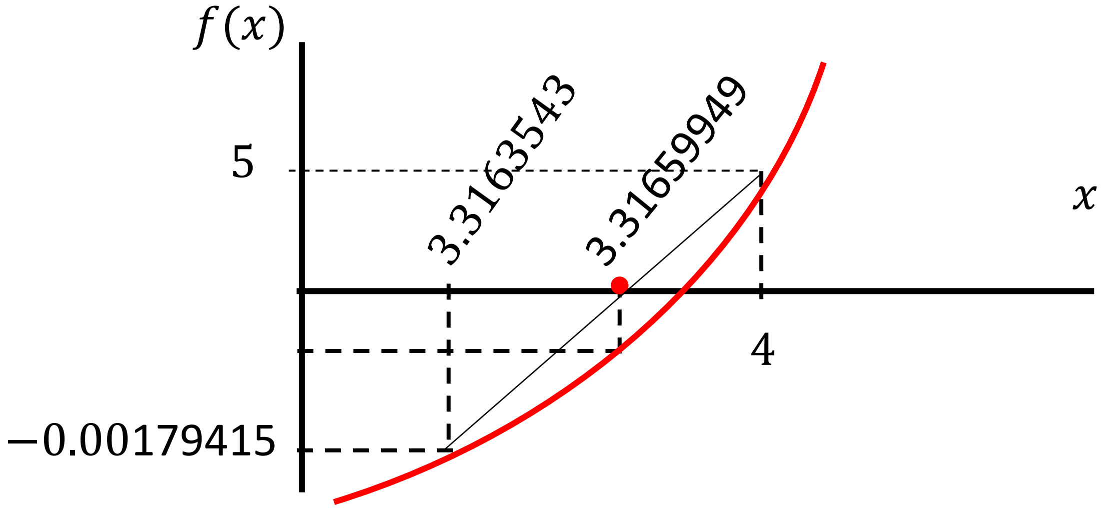
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.31659949$ and

$f(3.31659949) = 3.31659949^2 - 11 = -0.0001678$

$f(x_l) * f(x_r) = -0.00179415 * -0.0001678$

$\rightarrow 0.0000344961 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$=3.3163543 and CONTINUE to iterate.



Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l)f(x_r)$	$\lvert\varepsilon_a\rvert$ (%)
1	3	3.285714	4	-2	5	-0.2040816	0.4081632653	--
2	3.285714	3.313725	4	-0.20408	5	-0.0192234	0.0039231379	0.84531
3	3.313725	3.316354	4	-0.01922	5	-0.0017969	0.0000345424	0.07926
4	3.316354	3.316599	4	-0.0018	5	-0.0001678	0.0000003016	0.00741
5	3.316599	3.316622	4	-0.00017	5	-0.0000157	0.0000000026	0.00069
6	3.316622	3.316625	4	-1.6E-05	5	-0.0000015	0.0000000000	0.00006

- Note that false-position method converged faster than the bisection method.

Notes on Bracketing Methods

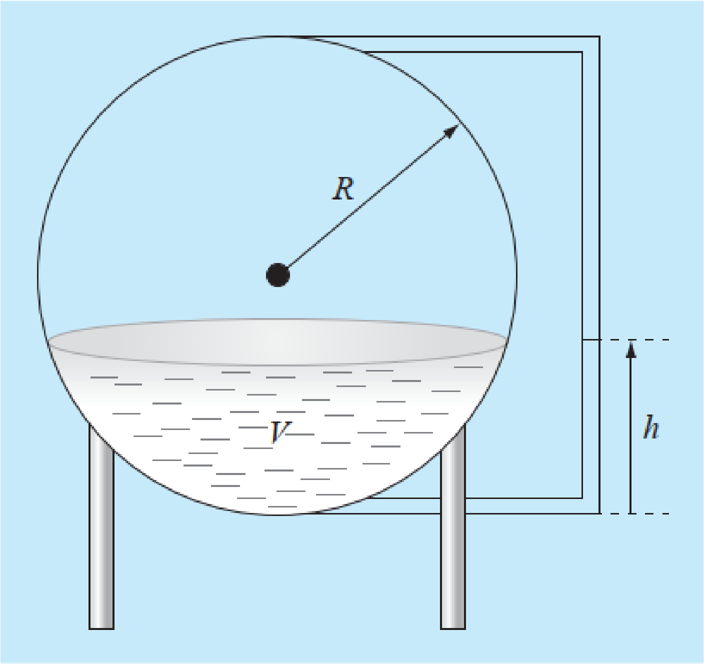
- A plot of the function is always helpful.
  - To determine the number of all roots, if there are any.
  - to determine whether the roots are multiple or not.
  - to determine whether to method converges to the desired root.
  - to determine the initial guesses.
- Incremental search technique can be used to determine the initial guesses.
  - Start from one end of the region of interest.
  - Evaluate the function at specified intervals.
  - If the sign of the function changes, than there is a root in that interval.
  - Select your intervals small, otherwise you may miss some of the roots. But if they are too small, incremental search might become too costly.
  - Incremental search, just by itself, can be used as a root finding technique with very small intervals (not efficient).

Problem 5.17

You are designing a spherical tank to hold water for a small village in a developing country. The volume of liquid it can hold can be computed as

$$V = \pi h^2 \frac{[3R - h]}{3}$$

where V = volume [m^3], h = depth of water in tank [m], and R= the tank radius [m]. If R = 3 m, to what depth must the tank be filled so that it holds 30 m^3? Use three iterations of the false-position method to determine your answer. Determine the approximate relative error after each iteration. Employ initial guesses of 0 and R.



NEXT WEEK
ROOTS OF EQUATIONS
Open Methods