# ME 209
# Numerical Methods

## 4. Solution of Linear Equation Systems:

## Part I

Asst. Prof. Dr. Nurettin Furkan DOĞAN

Mechanical Engineering Department

Gaziantep University

2024

# 4.1 INTRODUCTION

- In previous lecture, we determined the value $x$ that satisfied a single equation, $f(x) = 0$. Now, we deal with the case of determining the values $x_1, x_2, \ldots, x_n$ that simultaneously satisfy a set of equations

(General set of equations)

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ \cdot \qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad &\quad \cdot \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

- Such systems can be either *linear* or *nonlinear*. In this section, we deal with *linear algebraic equations* that are of the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdot \qquad\qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad\qquad &\quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where the $a$'s are constant coefficients, the $b$'s are constants, and $n$ is the number of equations.

- The system of linear equations given can be represented in matrix form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$[A]\{x\} = \{b\}$$

where $[A]$ is *n x n* Coefficient matrix
$\{x\}$ is *n x 1* Unknown vector
$\{b\}$ is *n x 1* Right-hand side (RHS) vector

- In this section, first some reminders about matrices will be made and the types of matrices and matrix operations will be briefly mentioned.
- Then, numerical methods used in solving sets of linear equations will be discussed.

# 4.2 Matrix

- A *matrix* consists of a rectangular array of elements represented by a single symbol.
- $[A]$ is the shorthand notation for the matrix and $a_{ij}$ designates an individual *element* of the matrix.
- A horizontal set of elements is called a *row* and a vertical set is called a *column*. The first subscript $i$ always designates the number of the row in which the element lies. The second subscript $j$ designates the column.

$$
\begin{array}{ccc}
 & \text{Column 3} & \\
[A] = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & \boxed{a_{23}} & \cdots & a_{2m} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} & \text{Row 2}
\end{array}
$$

*n x m* Matrix

**Row Vector:** Matrices with row dimension *n* = 1, such as

$$[B] = [b_1 \quad b_2 \quad \cdots \quad b_m]$$

**Column Vector:** Matrices with column dimension *n* = 1, such as

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

**Square Matrix:** Matrices where *n* = *m* are called *square matrices*. For example, a 4 by 4 matrix is

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The diagonal consisting of the elements $a_{11}$, $a_{22}$, $a_{33}$, and $a_{44}$ is termed the *principal* or *main diagonal* of the matrix.

# 4.2.1 Special Types of Square Matrix

- A ***symmetric matrix*** is one where $a_{ij} = a_{ji}$ for all $i$'s and $j$'s. For example, $[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$ is a 3 by 3 symmetric matrix.

- A ***diagonal matrix*** is a square matrix where all elements off the main diagonal are equal to zero, as in

$$[A] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{bmatrix}$$

Note that where large blocks of elements are zero, they are left blank.

- An ***identity matrix*** is a diagonal matrix where all elements on the main diagonal are equal to 1, as in

$$[A] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

The symbol $[I]$ is used to denote the identity matrix. The identity matrix has properties similar to unity.

- An ***upper triangular matrix*** is one where all the elements below the main diagonal are zero, as in

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix}$$

- An ***lower triangular matrix*** is one where all the elements above the main diagonal are zero, as in

$$[A] = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

- A ***banded matrix*** has all elements equal to zero, except for a band centered on the main diagonal:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$$

## 4.2.2 Matrix Operations

- ***Addition*** of two matrices, say, [$A$] and [$B$], is accomplished by adding corresponding terms in each matrix. The elements of the resulting matrix [$C$] are computed,

$$c_{ij} = a_{ij} + b_{ij} \qquad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m.$$

- Similarly, the ***subtraction*** of two matrices, say, [$E$] minus [$F$], is obtained by subtracting corresponding terms, as in

$$d_{ij} = e_{ij} - f_{ij} \qquad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m.$$

- Addition and subtraction can be performed only between matrices having the same dimensions.
- Both addition and subtraction are *commutative*:

$$[A] + [B] = [B] + [A]$$

- Addition and subtraction are also *associative*, that is,

$$([A] + [B]) + [C] = [A] + ([B] + [C])$$

- The ***multiplication*** of a matrix [*A*] by a scalar *g* is obtained by multiplying every element of [*A*] by *g*, as in

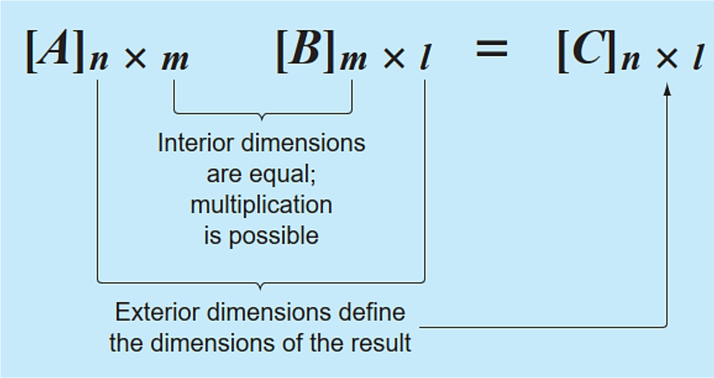
$$[D] = g[A] = \begin{bmatrix} ga_{11} & ga_{12} & \cdots & ga_{1m} \\ ga_{21} & ga_{22} & \cdots & ga_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ ga_{n1} & ga_{n2} & \cdots & ga_{nm} \end{bmatrix}$$

- The ***product*** of two matrices is represented as [*C*] = [*A*][*B*], where the elements of [*C*] are defined as

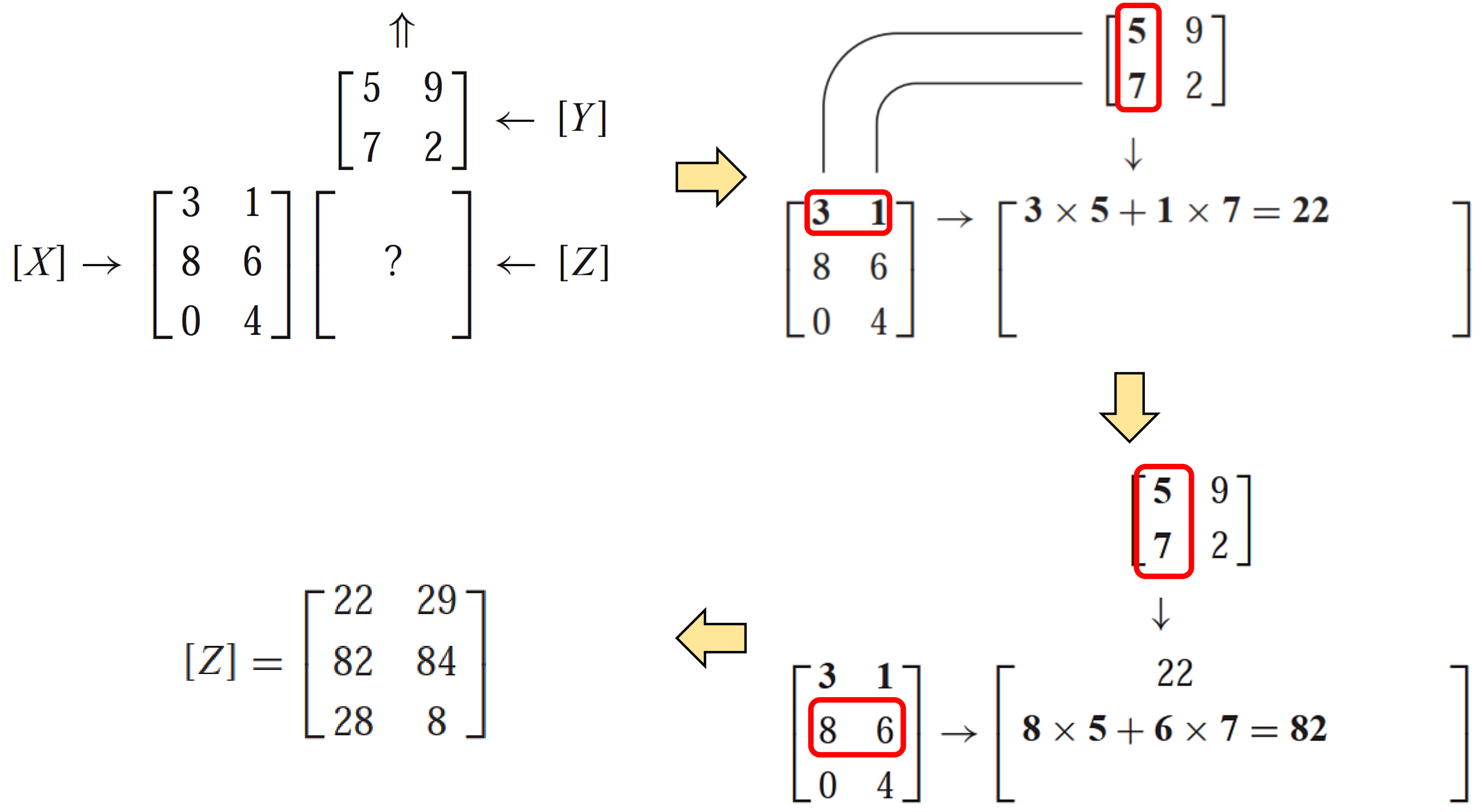
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where *n* = the column dimension of [*A*] and the row dimension of [*B*]. That is, the $c_{ij}$ element is obtained by adding the product of individual elements from the *i*th row of the first matrix, in this case [*A*], by the *j*th column of the second matrix [*B*].

- According to this definition, multiplication of two matrices can be performed *only if the first matrix has as many columns as the number of rows in the second matrix.*



- Suppose that we want to multiply [$X$] by [$Y$] to yield [$Z$], $[Z] = [X][Y] = \begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix}$
- A simple way to visualize the computation of [$Z$] is to raise [$Y$], as in



- If the dimensions of the matrices are suitable, matrix multiplication is *associative*,

$$([A][B])[C] = [A]([B][C])$$

- and *distributive*,

$$[A]([B] + [C]) = [A][B] + [A][C]$$

or

$$([A] + [B])[C] = [A][C] + [B][C]$$

- However, multiplication is *not generally commutative*:

$$[A][B] \neq [B][A]$$

- The ***transpose of a matrix*** involves transforming its rows into columns and its columns into rows.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \Rightarrow [A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

- In other words, the element $a_{ij}$ of the transpose is equal to the $a_{ji}$ element of the original matrix.
- The ***trace*** of a matrix is the sum of the elements on its principal diagonal. It is designated as tr $[A]$ and is computed as

$$\text{tr}\ [A] = \sum_{i=1}^{n} a_{ii}$$

- If a matrix $[A]$ is square and nonsingular, there is another matrix $[A]^{-1}$, called the ***inverse*** of $[A]$, for which

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

- The *inverse of a two-dimensional square matrix* can be represented simply by

$$[A]^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- The ***determiant of a matrix*** is equal to the sum of the products of all elements in any row or column by their cofactors.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2N} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3i} & \cdots & a_{3j} & \cdots & a_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ii} & \cdots & a_{ij} & \cdots & a_{iN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{ji} & \cdots & a_{jj} & \cdots & a_{jN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{Ni} & \cdots & a_{Nj} & \cdots & a_{NN} \end{vmatrix} = \sum_{i=1}^{N} a_{ik} M_{ik} (-1)^{i+k} = \sum_{j=1}^{N} a_{kj} M_{kj} (-1)^{k+j}$$

Cofactor matrix *M is* the matrix composed of multiplication of the minors of *A* by $(-1)^{i+j}$:

*Example:* Calculate the determinant and inverse of matrix A.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

We need the cofactor matrix $C$ of $A$ to find the inverse and determinant of matrix $A$:

$$C = \begin{bmatrix} \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\det(A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 7 - 3 - 3 = 1$$

(Using 1st column elements)

$$A^{-1} = C^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- The final matrix manipulation that will have utility in our discussion is ***augmentation***. A matrix is augmented by the addition of a column (or columns) to the original matrix.
- For example, suppose that matrix *A* augmented with the column matrix *B*:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \vdots & b_{11} \\ a_{21} & a_{22} & \vdots & b_{21} \end{bmatrix}$$

# 4.3 Solving a Small ($n \leq 3$) Set of Equations

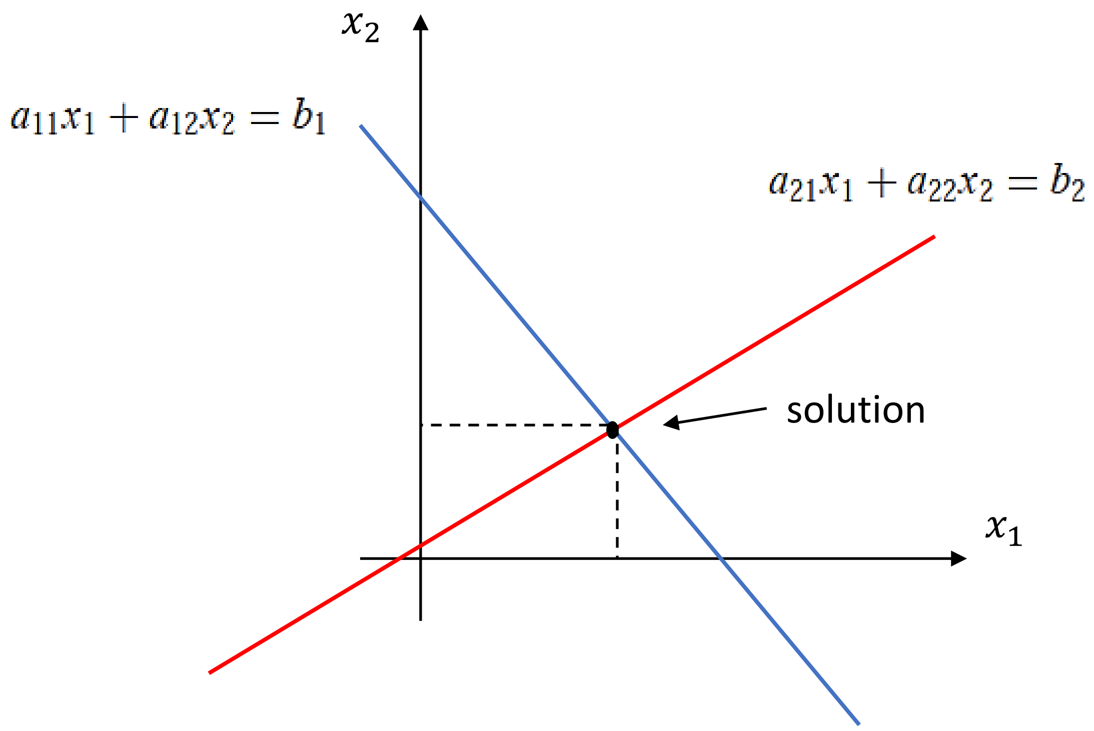
## 4.3.1 Graphical Method

- Consider a set of 2 equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Both equations can be solved for $x_2$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}}$$
$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



**Example:** Use the graphical method to solve

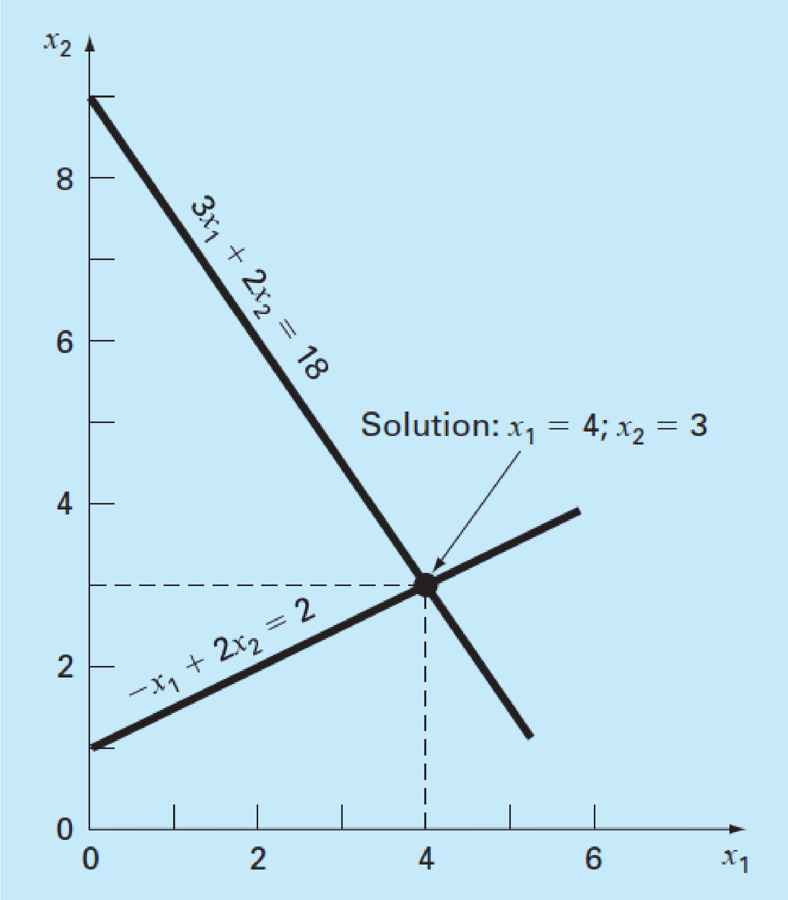
$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

- Solve equations for $x_2$ and plot in $x_1$ - $x_2$ axes.

$$3x_1 + 2x_2 = 18 \longrightarrow x_2 = -\frac{3}{2}x_1 + 9$$

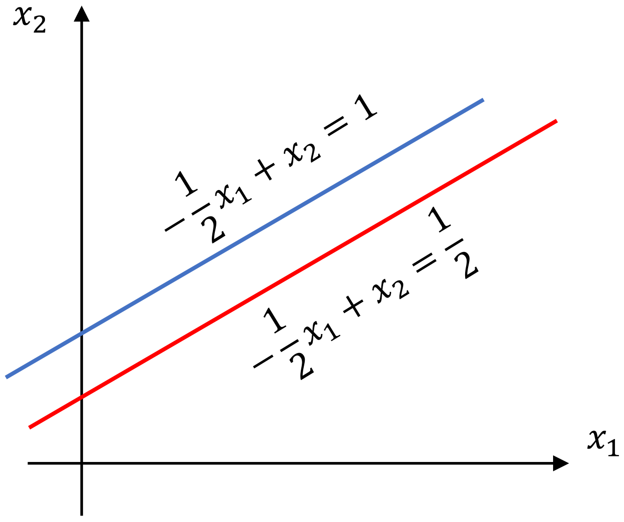
$$-x_1 + 2x_2 = 2 \longrightarrow x_2 = \frac{1}{2}x_1 + 1$$

- The solution is the intersection of the two lines at $x_1$=4 and $x_2$=3.



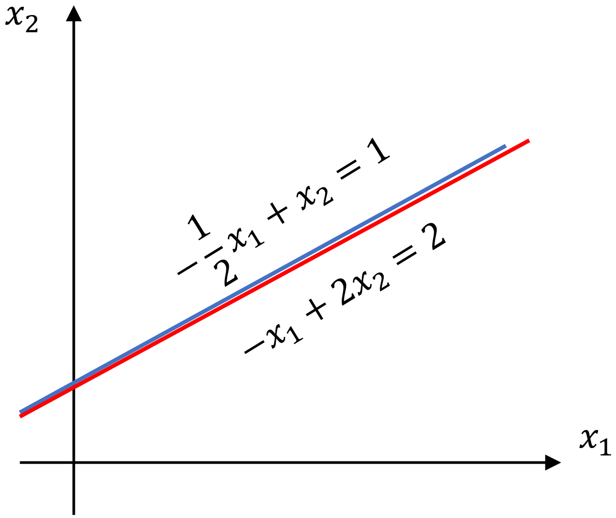

- For $n$=3, each equation will be a plane on a 3D coordinate system. Solution is the point where these planes intersect.
- • For $n$>3, graphical solution is not practical.

- Following figures represent the case that the graphical methods break down.



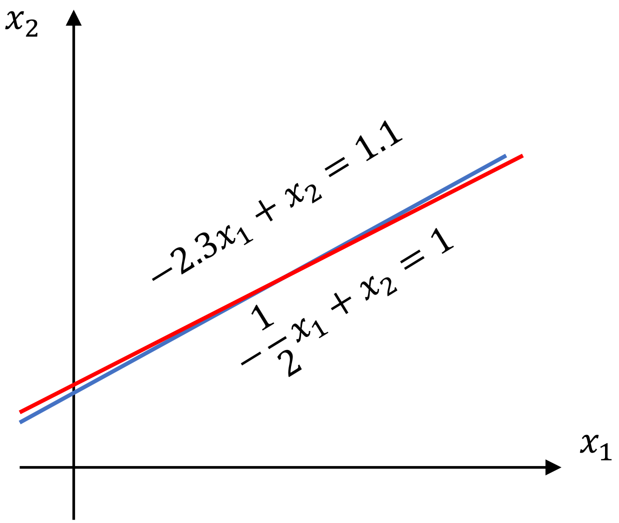

$(a)$

Parallel lines = no solution




$(b)$

Coinciding lines = infinite solutions




$(c)$

Very close to being singular = hard to identify the solution

# 4.3.2 Cramer's Rule

- Determinant of 2 by 2 system: $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

- Determinant of 3 by 3 system: $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

*minors*

***Cramer's Rule:*** Each unknown is calculated as a fraction of two determinants. The denominator is the determinant of the system, D. The numerator is the determinant of a modified system obtained by replacing the column of coefficients of the unknown being calculated by the right-hand-side (RHS) vector.

## 4.3.2 Cramer's Rule

For a *3x3* system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \longrightarrow [A]\{x\} = \{b\}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

**Example:** Use the Cramer's rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

The determinant $D$ can be written as

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} = 0.3(-0.07) - 0.52(0.06) + 1(0.05) = -0.0022$$

The minors are

$$A_1 = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 1(0.5) - 1.9(0.3) = -0.07$$

$$A_2 = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.5(0.5) - 1.9(0.1) = 0.06$$

$$A_3 = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.5(0.3) - 1(0.1) = 0.05$$

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

## 4.3.3 Naive Gauss Elimination Method

- The approach is designed to solve a general set of n equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

**STEP 0 (Optional):** Form the augmented matrix of $[A|B]$.

**STEP 1 Forward Elimination:** Reduce the system to an *upper triangular system*.

1. The initial step will be to eliminate the first unknown, $x_1$, from the second through the *n*th equations.
2. Multiply the *1*st eqn. by $a_{21}/a_{11}$ & subtract it from the *2*nd equation. This is the new *2*nd eqn.
3. ...
4. Multiply the *1*st eqn. by $a_{n1}/a_{11}$ & subtract it from the *n*th equation. This is the new *n*th eqn.

**Important:** In these steps the 1*st* eqn is the **pivot equation** and $a_{11}$ is the **pivot element**. Note that a division by zero may occur if the pivot element is zero. Naive-Gauss Elimination does not check for this.

The modified system is
$$\begin{bmatrix} a_{11} & a_{11} & a_{11} & \dots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \dots & a'_{3n} \\ \dots & \dots & \dots & \ddots & \vdots \\ 0 & a'_{n2} & a'_{n3} & \dots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{Bmatrix}$$

$'$ indicates that the system is modified once.

- Eliminate $x_2$ from *3rd* to *nth* equations

The modified system is
$$\begin{bmatrix} a_{11} & a_{11} & a_{11} & \dots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} \\ 0 & 0 & a''_{33} & \dots & a''_{3n} \\ \dots & \dots & \dots & \ddots & \vdots \\ 0 & a''_{n2} & a''_{n3} & \dots & a2'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b''_n \end{Bmatrix}$$

$''$ indicates that the system is modified once.

- Repeat the last two procedure to unkonwns $x_3$ *to* $x_n$eliminated from *4th* to *nth* equations

At the end of the step 1
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{Bmatrix}$$

Primes are removed for clarity.

**STEP 2 Back Substitution:** Find the unknowns starting from the last equation.

- Last equation involves only $x_n$. Solve for it.
- Use this $x_n$ in the *(n-1)*th equation and solve for $x_{n-1}$.
- ...
- Use all previously calculated *x* values in the *1*st eqn and solve for $x_1$.

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$$

$$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$$

$$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$$

$$-6x_1 + 4x_2 + x_3 - 18x_4 = -34$$

**Example:** Use gauss elimination to solve the following system of equations.

**Step 0:** Construct augmented matrix

$$\begin{bmatrix} 6 & -2 & 2 & 4 & | & 16 \\ 12 & -8 & 6 & 10 & | & 26 \\ 3 & -13 & 9 & 3 & | & -19 \\ -6 & 4 & 1 & -18 & | & -34 \end{bmatrix}$$

**Step 1:** Forward elimination

$\boxed{\phantom{0}}$ : Pivot element

Eliminate $x_1$

$$\left[\begin{array}{cccc|c} \boxed{6} & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right]$$

- $-2 * R_1 + R_2$
- $-\frac{1}{2} * R_1 + R_3$
- $R_1 + R_4$

$$\longrightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \end{array}\right]$$

Eliminate $x_2$

$$\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & \boxed{-4} & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \end{array}\right]$$

- $-3 * R_2 + R_3$
- $\frac{1}{2} * R_2 + R_4$

$$\longrightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right]$$

Eliminate $x_3$

$$\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & \boxed{2} & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right]$$

- $-2 * R_3 + R_4$

$$\longrightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right]$$

**Step 2:** Back substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{Bmatrix}$$

$$-3x_4 = -3 \rightarrow \boxed{\boldsymbol{x_4 = 1}}$$

$$2x_3 - 5x_4 = -9 \rightarrow \boxed{\boldsymbol{x_3 = -2}}$$

$$-4x_2 + 2x_3 + 2x_4 = -6 \rightarrow \boxed{\boldsymbol{x_2 = 1}}$$

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \rightarrow \boxed{\boldsymbol{x_1 = 3}}$$

# PIVOTING

- ***Pivoting*** is the displacement of rows in the coefficient matrix so that the diagonal elements are maximized in absolute value.
- Pivoting is employed to prevent division by zero, a pitfall that could cause the failure of the Naive Gauss elimination method.

**Example:** Solve the following system using Gauss Elimination with pivoting.

$$2x_2 + x_4 = 0$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$4x_1 - 3x_2 + x_4 = -7$$
$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

**Step 0:** Construct augmented matrix

$$\begin{bmatrix} 0 & 2 & 0 & 1 & | & 0 \\ 2 & 2 & 3 & 2 & | & -2 \\ 4 & -3 & 0 & 1 & | & -7 \\ 6 & 1 & -6 & -5 & | & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 & 1 & | & 0 \\ 2 & 2 & 3 & 2 & | & -2 \\ 4 & -3 & 0 & 1 & | & -7 \\ 6 & 1 & -6 & -5 & | & 6 \end{bmatrix} \xrightarrow{\text{Pivoting } R_1 \text{ and } R_4} \begin{bmatrix} 6 & 1 & -6 & -5 & | & 6 \\ 2 & 2 & 3 & 2 & | & -2 \\ 4 & -3 & 0 & 1 & | & -7 \\ 0 & 2 & 0 & 1 & | & 0 \end{bmatrix}$$

**Step 1:** Forward elimination

Eliminate $x_1$

$$\begin{bmatrix} \boxed{6} & 1 & -6 & -5 & | & 6 \\ 2 & 2 & 3 & 2 & | & -2 \\ 4 & -3 & 0 & 1 & | & -7 \\ 0 & 2 & 0 & 1 & | & 0 \end{bmatrix}$$

- $-\frac{1}{3} * R_1 + R_2$
- $-\frac{2}{3} * R_1 + R_3$

$$\rightarrow \begin{bmatrix} 6 & 1 & -6 & -5 & | & 6 \\ 0 & 1.6667 & 5 & 3.6667 & | & -4 \\ 0 & -3.6667 & 4 & 4.3333 & | & -11 \\ 0 & 2 & 0 & 1 & | & 0 \end{bmatrix}$$

Pivoting $R_2$ and $R_3$. Then, eliminate $x_2$

$$\begin{bmatrix} 6 & 1 & -6 & -5 & | & 6 \\ 0 & -3.6667 & 4 & 4.3333 & | & -11 \\ 0 & 1.6667 & 5 & 3.6667 & | & -4 \\ 0 & 2 & 0 & 1 & | & 0 \end{bmatrix}$$

- $\frac{1.6667}{3.6667} * R_2 + R_3$
- $\frac{2}{3.6667} * R_2 + R_4$

$$\rightarrow \begin{bmatrix} 6 & 1 & 6 & -5 & | & 6 \\ 0 & -3.6667 & 4 & 4.3333 & | & -11 \\ 0 & 0 & 6.8182 & 5.6364 & | & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & | & -5.9999 \end{bmatrix}$$

Eliminate $x_3$

$$\left[\begin{array}{cccc|c} 6 & 1 & 6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & -5.9999 \end{array}\right] \quad \bullet \ \frac{-2.1818}{6.8182} * R_3 + R_4 \rightarrow \left[\begin{array}{cccc|c} 6 & 1 & 6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{array}\right]$$

**Step 2:** Backward substitution

$$1.56x_4 = -3.1199 \rightarrow \boxed{\boldsymbol{x_4 = -1.9999}}$$

$$6.8182x_3 + 5.6364x_4 = -9.0001 \rightarrow \boxed{\boldsymbol{x_3 = 0.33325}}$$

$$-3.6667x_2 + 4x_3 + 4.3333x_4 = -11 \rightarrow \boxed{\boldsymbol{x_2 = 1.0}}$$

$$6x_1 + x_2 + 6x_3 - 5x_4 = 6 \rightarrow \boxed{\boldsymbol{x_1 = -0.5}}$$

# SCALING

- ***Scaling*** is to normalize the equations so that the maximum coefficient in every row is equal to 1.0. That is, divide each row by the coefficient in that row with the maximum magnitude.
- It is advised to scale a system before calculating its determinant. This is especially important if we are calculating the determinant to see if the system is ill-conditioned or not.
- Consider the following systems

$$\begin{aligned} 2x_1 - 3x_2 &= 5 \\ 3.98x_1 - 6x_2 &= 7 \end{aligned} \qquad\qquad \begin{aligned} 20x_1 - 30x_2 &= 50 \\ 39.8x_1 - 60x_2 &= 70 \end{aligned}$$

- They are actually the same system. In the second one the equations are multiplied by 10.
- Determinant of the 1st system is 2(-6) – (-3)(3.98) = -0.06 , which is close to zero.
- Determinant of the 2nd system is 20(-60) – (-30)(39.8) = -6 , which is not that close to zero.
- So, is this system ill-conditioned or not?

## 4.3.4 LU Decomposition Method

The Gauss elimination method can essentially be expressed as follows using matrix notation.

$$\boldsymbol{A}.x = \boldsymbol{B}$$
$$\boldsymbol{A} = \boldsymbol{P}.\boldsymbol{L}.\boldsymbol{U}$$

Here, $\boldsymbol{P}$ is the matrix expressing the row displacements during pivoting, $\boldsymbol{L}$ is the lower triangular matrix consisting of the multipliers used during the zeroing of the columns, and $\boldsymbol{U}$ is the upper triangular matrix reached in the Gauss elimination method.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 & \dots & 0 \\ L_{21} & L_{22} & 0 & \dots & 0 \\ L_{31} & L_{32} & L_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & \dots & U_{1n} \\ 0 & 1 & U_{23} & \dots & U_{2n} \\ 0 & 0 & 1 & \dots & U_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 & \dots & 0 \\ L_{21} & L_{22} & 0 & \dots & 0 \\ L_{31} & L_{32} & L_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & \dots & U_{1n} \\ 0 & 1 & U_{23} & \dots & U_{2n} \\ 0 & 0 & 1 & \dots & U_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

By multiplying matrix ***L*** with matrix ***U*** and equalizing it to the corresponding element of ***A***, the elements of matrices ***L*** and ***U*** are found as follows:

$$a_{11} = L_{11}.1 + 0.0 + 0.0 + \cdots \quad \rightarrow \quad L_{11} = a_{11}$$

$$a_{21} = L_{21}.1 + L_{22}.0 + 0.0 + \cdots \quad \rightarrow \quad L_{21} = a_{21}$$

................

$$a_{n1} = L_{n1}.1 + L_{n2}.0 + L_{n3}.0 + \cdots \quad \rightarrow \quad L_{n1} = a_{n1}$$

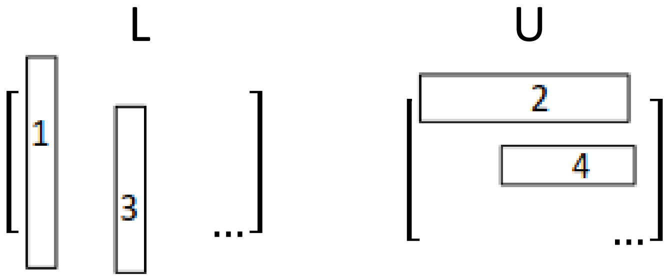
$$a_{12} = L_{11}.U_{12} + 0.1 + 0.0 + \cdots \quad \rightarrow \quad U_{12} = \frac{a_{12}}{L_{11}}$$

$$a_{13} = L_{11}.U_{13} + 0.U_{23} + 0.1 + \cdots \quad \rightarrow \quad U_{13} = \frac{a_{13}}{L_{11}}$$

................

$$a_{1n} = L_{11}.U_{1n} + 0.U_{2n} + 0.U_{3n} + \cdots \quad \rightarrow \quad U_{1n} = \frac{a_{1n}}{L_{11}}$$

Note that, the order given below is followed to find the matrix elements.



In the 2nd step, after the ***L*** and ***U*** matrices are found, first the intermediate value vector $\vec{y}$ and then the unknown vector $\vec{x}$ are found by the following operations.

$$L.\vec{y} = \vec{b} \quad \Longrightarrow \quad U.\vec{x} = \vec{y}$$

**Example:** Solve the following system of equations using the LU method.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= -3 \\ 2x_1 + 5x_2 + 2x_3 &= -8 \\ 3x_1 + x_2 + 5x_3 &= 1 \end{aligned}$$

First, let's write the coefficient matrix and decompose to LU matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$L_{11} = 1$ $\qquad$ $L_{21} = 2$ $\qquad$ $L_{31} = 3$

$L_{11} * U_{12} = 2 \rightarrow U_{12} = 2$

$L_{21} * U_{12} + L_{22} = 5 \rightarrow L_{22} = 1$

$L_{31} * U_{12} + L_{32} = 1 \rightarrow L_{32} = -5$

$L_{11} * U_{13} = 3 \rightarrow U_{13} = 3$

$L_{21} * U_{13} + L_{22} * U_{23} = 2 \rightarrow U_{23} = -4$

$L_{31} * U_{13} + L_{32} * U_{23} + L_{33} = 5 \rightarrow L_{33} = -24$

Second, vector $\vec{y}$ according to the expression $L.\vec{y} = \vec{b}$ with forward sweep:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & -24 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 1 \end{bmatrix} \Longrightarrow \begin{aligned} y_1 &= -3 \\ y_2 &= -2 \\ y_3 &= 0.0 \end{aligned}$$

Using these values, the solution vector is found from the $U.\vec{x} = \vec{y}$ equation by the back sweep method.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0.0 \end{bmatrix} \Longrightarrow \boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 0.0 \end{aligned}}$$

## 4.3.5 Gauss-Jordan Method

- This method is similar to the Gauss elimination method and has two stages.
- In the first stage, the coefficients matrix (*A*) of the given system of linear equations is made diagonal by basic row operations.
- That is, the elements both below and above the matrix diagonal are reset to zero, and the diagonal element values are set to *1*, as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & b'_1 \\ 0 & 1 & 0 & \dots & 0 & b'_2 \\ 0 & 0 & 1 & \dots & 0 & b'_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & b'_n \end{bmatrix}$$

**Example:** Solve the following system of equations by Gauss-Jordan Method.

$$2x_2 + x_4 = 0$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$4x_1 - 3x_2 + x_4 = -7$$
$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

Pivoting

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array}\right] \quad R_1 \leftrightarrow R_4 \quad \left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Divide 1st row ($R_1$) by the diagonal element 6: $(R_1/6)$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad R_1/6 \quad \left[\begin{array}{cccc|c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Eliminate $x_1$ from 2nd to 4th equations:

$$\left[\begin{array}{cccc:c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{c} R_1 * (-2) + R_2 \\ R_1 * (-4) + R_3 \end{array} \quad \left[\begin{array}{cccc:c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3 \quad \left[\begin{array}{cccc:c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Divide 2nd row ($R_2$) by the diagonal element -3.6667:

$$\left[\begin{array}{cccc:c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad R_2/(-3.6667) \quad \left[\begin{array}{cccc:c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Eliminate $x_2$ from 1st to 4th equations:

$$\left[\begin{array}{cccc|c} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{c} R_2 * (-0.1667) + R_1 \\ R_2 * (-1.6667) + R_3 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & -0.8182 & -.0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 6.8182 & 5.6364 & -9 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{array}\right]$$

Divide 3rd row ($R_3$) by the diagonal element 6.8182:

$$\left[\begin{array}{cccc|c} 1 & 0 & -0.8182 & -.0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 6.8182 & 5.6364 & -9 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{array}\right] \quad R_3/(6.8182) \quad \left[\begin{array}{cccc|c} 1 & 0 & -0.8182 & -.0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{array}\right]$$

Eliminate $x_3$ from 1st to 4th equations:

$$\left[\begin{array}{cccc|c} 1 & 0 & -0.8182 & -.0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{array}\right] \quad \begin{array}{c} R_3 * (1.0909) + R_2 \\ R_3 * (-2.1818) + R_4 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.0400 & -0.58 \\ 0 & 1 & 0 & -0.2800 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1.5600 & -3.12 \end{array}\right]$$

Divide 4th row ($R_4$) by the diagonal element 1.5600:

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0.0400 & -0.58 \\ 0 & 1 & 0 & -0.2800 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1.5600 & -3.12 \end{array}\right] \quad R_4/(1.5600) \quad \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0.0400 & -0.58 \\ 0 & 1 & 0 & -0.2800 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Eliminate $x_4$ from 1st to 3rd equations:

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0.0400 & -0.58 \\ 0 & 1 & 0 & -0.2800 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{array}{c} R_4 * (-0.0400) + R_1 \\ R_4 * (0.2800) + R_2 \\ R_4 * (-0.8267) + R_3 \end{array} \quad \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 1. \\ 0 & 0 & 1 & 0 & 0.3333 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Solution:

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 1. \\ 0 & 0 & 1 & 0 & 0.3333 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\boxed{\begin{array}{c} x_1 = -0.5 \\ x_2 = 1 \\ x_3 = 0.3333 \\ x_4 = -2 \end{array}}$$

NEXT WEEK
Solutions of Linear Equation Systems
Part II