Appendix A
AREA MOMENTS OF INERTIA

APPENDIX OUTLINE

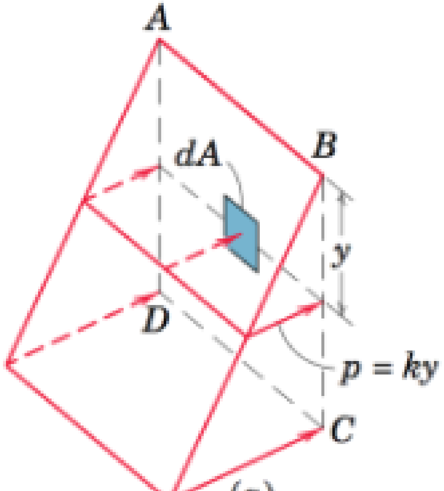
A/1 Introduction
A/2 Definitions
A/3 Composite Areas

Article A/1 Introduction

- When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area.
- Frequently the intensity of the force (pressure or stress) is proportional to the distance of the line of action of the force from the moment axis.
- The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area.
- We see, therefore, that the total moment involves an integral of form:

$$Moment\ of\ Inertia\ (I) = \int (distance)^2\, d(area)$$

Article A/1 Introduction



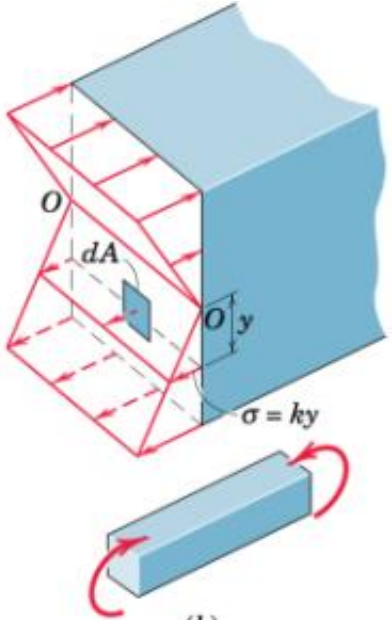
(a)

- The moment of the liquid pressure p on dA about line AB is:

$$pydA = \mathrm{ky^2dA}$$

- Total moment:

$$M = \mathrm{k}\int \mathrm{y^2dA}$$



(b)

- The moment of stress σ on dA about line OO is:

$$y(\sigma dA) = \mathrm{ky^2dA}$$

- Total moment:

$$M = \mathrm{k}\int \mathrm{y^2dA}$$

Article A/1 Introduction

- Although the integral illustrated in the preceeding examples is generally called the *moment of inertia*, a more correct term is ***the second moment of area***, since the first moment ydA is multiplied by the moment arm y to obtain the second moment for the element dA.
- The moment of inertia of an area is a purely mathematical property of the area and in itself has no physical significance.

Article A/2 Definitions

- **Rectangular and Polar Moment of Inertia**
- Consider the area *A* in the *x*-*y* plane. The moments of inertia of the element *dA* about the *x*- and *y*- axes are,

$$dI_x = y^2 dA \text{ and } dI_y = x^2 dA$$

The moments of inertia of *A* about same axes are :

$$I_x = \int y^2 dA$$

$$I_y = \int x^2 dA$$

(Rectangular moments of inertia)

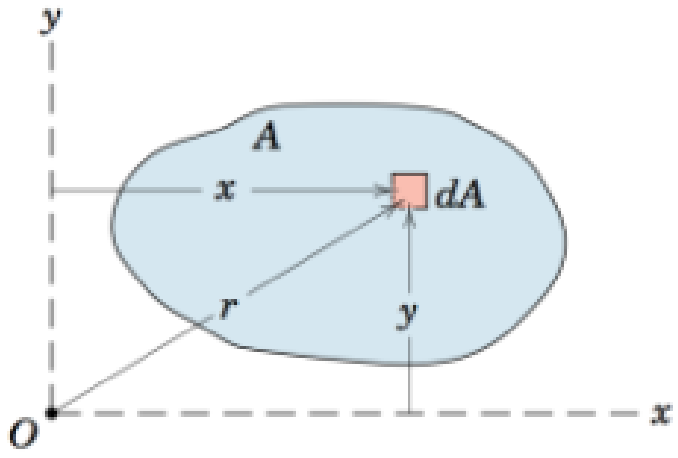


Figure A/2

Article A/2 Definitions

- **Rectangular and Polar Moment of Inertia**
- The moment of inertia of dA about the pole O (z-axis) is

$dI_z = r^2 dA$. The moment of inertia of the eintire area about O is:

$$I_z = \int r^2 dA$$

(Polar moment of inertia)

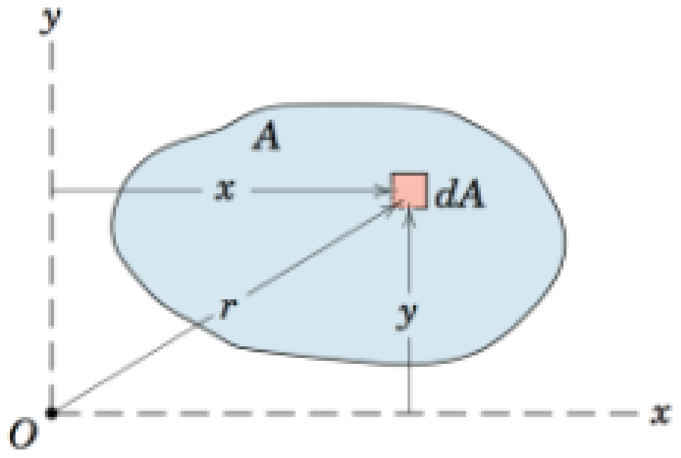


Figure A/2

Due to $x^2 + y^2 = r^2$

$$I_z = I_x + I_y$$

Article A/2 Definitions

- The moment of inertia of an element involves the square of the distance from the inertia axis to the element.
- Thus ***an element having negative coordinate contributes as much the the moment of inertia as does an equal element with positive coordinate of the same magnitude***.
- Consequently, ***the area moment of inertia about any axis*** is **always** a ***positive quantity***.
- In contrast, the first moment of area, which was involved in the computations of centroids, could be either positive, negative, or zero.
- The SI units for the area moments of inertia are expressed as quadric meter (m^4) or quadric milimeters (mm^4).

Article A/2 Definitions

- The choice of the coordinates to use for moments of inertia is quite important. Rectangular coordinates should be used for shapes whose boundaries are most easily expressed in these coordinates.
- Polar coordinates will usually simplify problems involving boundaries which are easily described in r and θ.
- The choice of element of area which simplifes the integration as much as possible is also important. These considerations are quite similar to those we discussed for calculations of centroid.

Article A/2 Definitions

- **Radius of Gyration**

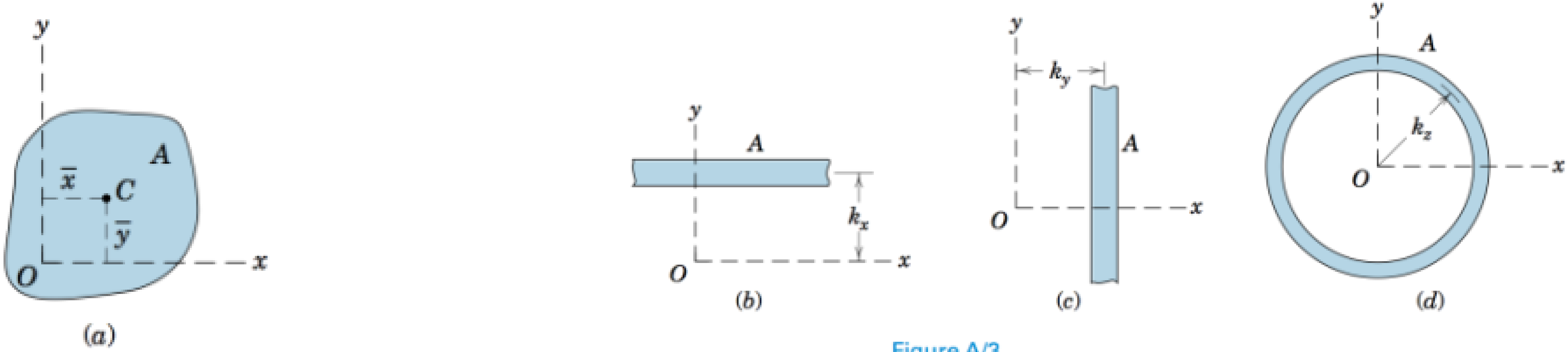


Figure A/3

- Consider the area A which has rectangular moments of inertia I_x and I_y and polar moment of inertia I_z about O.
- Now, let's consider this area as concentrated into a long narrow strip pf area A a distance k_x from the x- axis.

Article A/2 Definitions

- **Radius of Gyration**

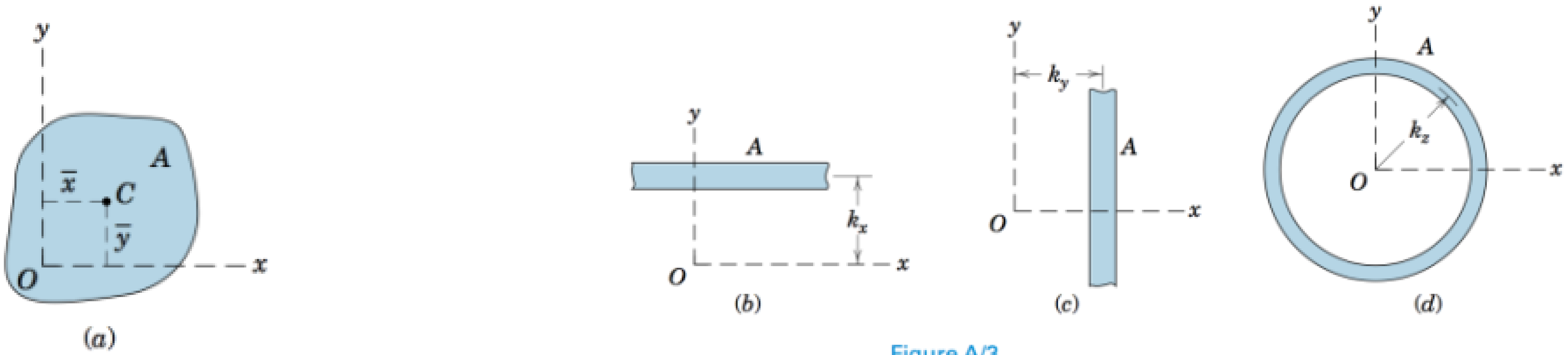


Figure A/3

- The moment of inertia of the strip about the *x*-axis

$$I_x = k_x{}^2 A$$

The distance k_x is called the *Radius of gyration* of the area about the *x*-axis.

Article A/2 Definitions

- **Radius of Gyration**

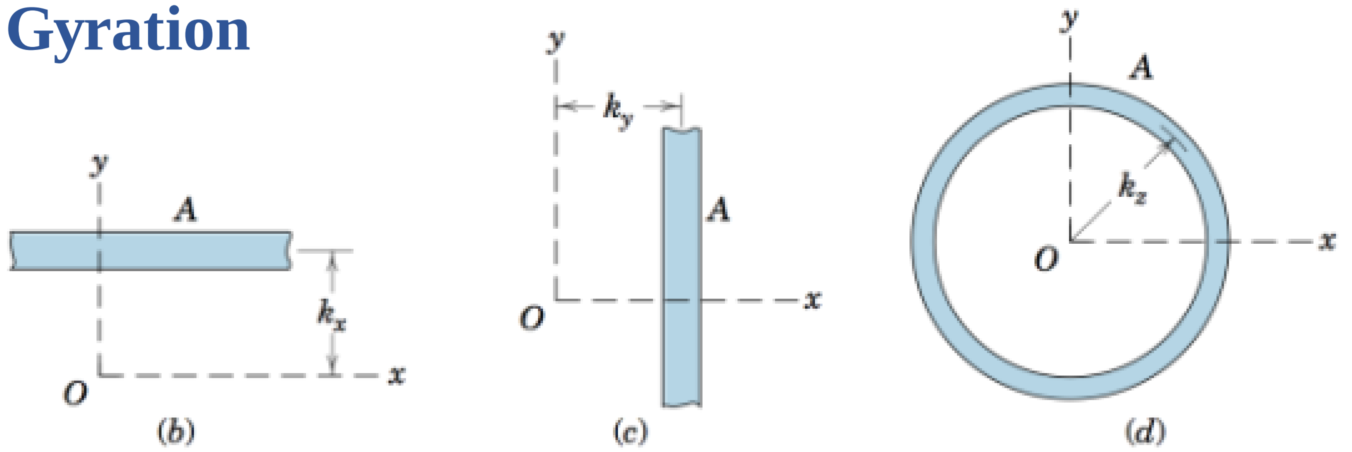


Figure A/3

- In summary:

$$I_x = \int k_x^{\,2} dA$$

$$I_y = \int k_y^{\,2} dA$$

$$I_z = \int k_z^{\,2} dA$$

or

$$k_x = \sqrt{I_x/A}$$

$$k_y = \sqrt{I_y/A}$$

$$k_z = \sqrt{I_z/A}$$

Article A/2 Definitions

- **Radius of Gyration**
- The relation between rectangular and polar radii of gyration can be formulated as:

$$k_z^2 = k_x^2 + k_y^2$$

Article A/2 Definitions

- **Transfer of Axes (Parallel Axis Theorem)**
- The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis.
- In figure the x_0-y_0 axes pass through the centroid C of the area.
- The moments of inertia of the element dA about the parallel x-y axes:

$$dI_x = (y_0 + d_x)^2 dA$$

- Integrating the equation above:

$$I_x = \int y_0^2 dA + 2d_x \int y_0 dA + d_x^2 \int dA$$

$$I_x = \bar{I}_x + Ad_x^2$$

$$I_y = \bar{I}_y + Ad_y^2$$

$$I_z = \bar{I}_z + Ad^2$$

The second integral is zero, since $\int y_0 dA = y_0 A$ and y_0 is automatically zero with the centroid on the x_0-axis.

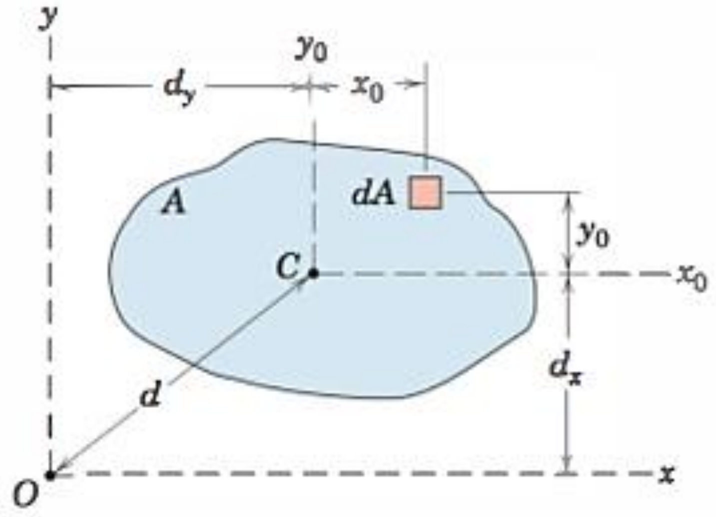


Figure A/4

Article A/2 Definitions

- **Transfer of Axes (Parallel Axis Theorem)**
- Please noted that:
- it is necessary to transfer from one axis to the parallel centroidal axis, then to transfer from the centroidal axis to the second axis.

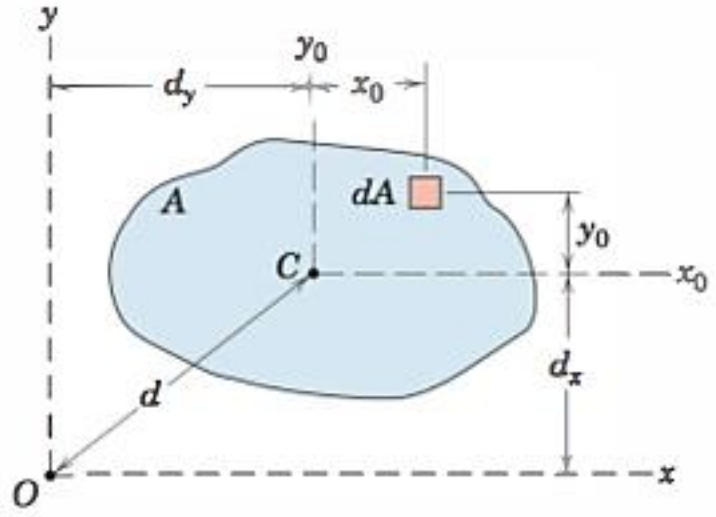
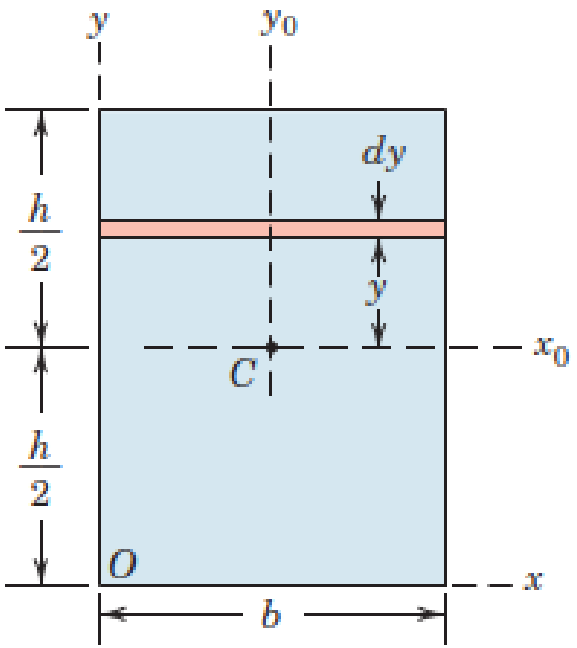


Figure A/4

Sample Problem A/1

- Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Sample Problem A/1

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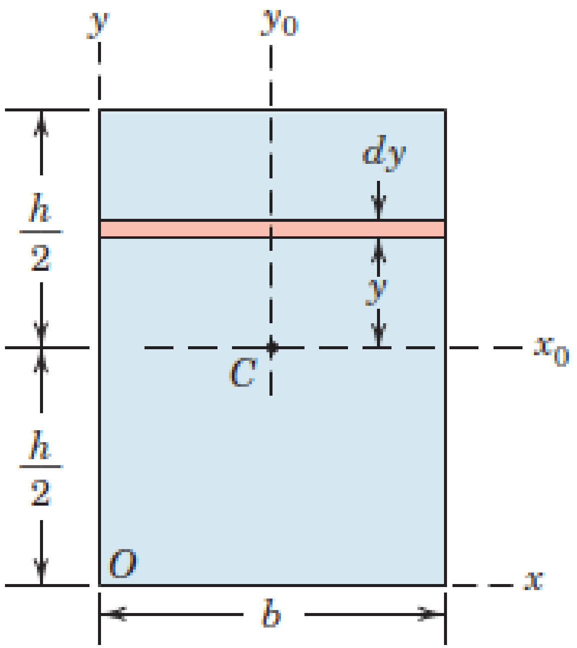


For the calculation of the moment of inertia about the x_0-axis, a horizontal strip of area bdy is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$\left[I_x = \int y^2\, dA\right] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\, dy = \frac{1}{12} bh^3$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

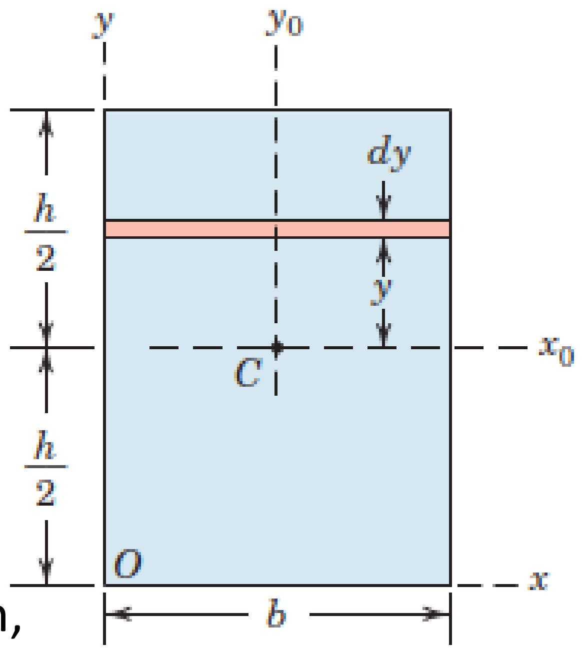
$$\bar{I}_y = \frac{1}{12} hb^3$$

The centroidal polar moment of inertia is

$$\left[\bar{I}_z = \bar{I}_x + \bar{I}_y\right] \qquad \bar{I}_z = \frac{1}{12}(bh^3 + hb^3) = \frac{1}{12} A(b^2 + h^2)$$

Sample Problem A/1

- Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



By the parallel-axis theorem the moment of inertia about the *x*-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}] \qquad I_x = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 = \frac{1}{3}Ah^2$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

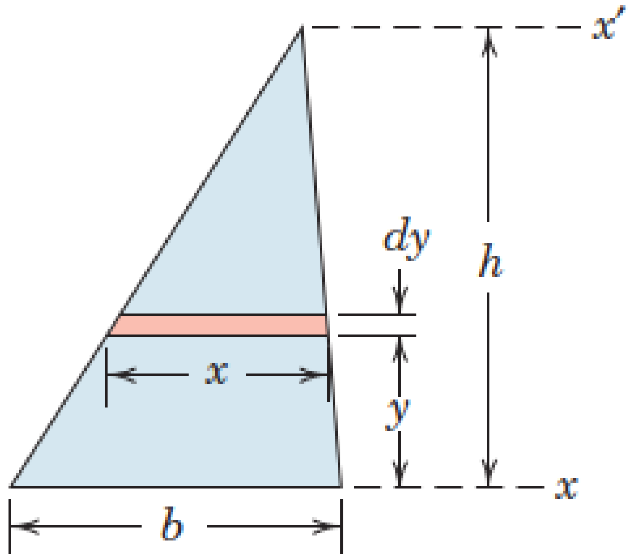
$$[I_z = \bar{I}_z + Ad^2]$$

$$I_z = \frac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \frac{1}{3}A(b^2 + h^2)$$

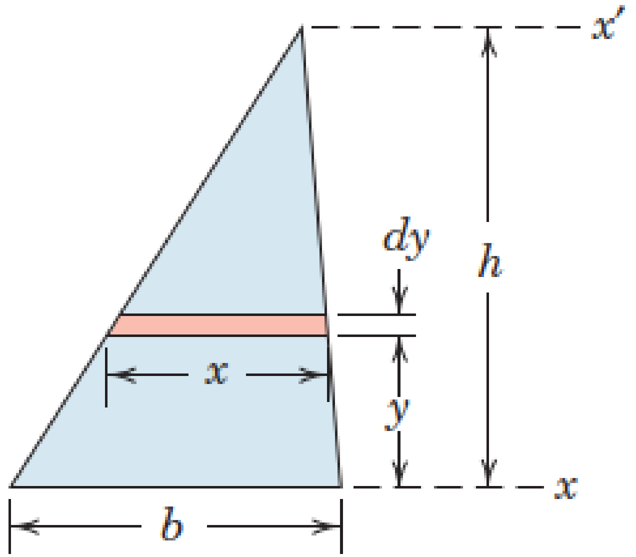
Sample Problem A/2

- Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Sample Problem A/2

- Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Area of the horizontal strip: $dA = xdy = \left[\frac{(h-y)b}{h}\right]dy$

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\, dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12}$$

By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance *h/3* above the *x*-axis, is

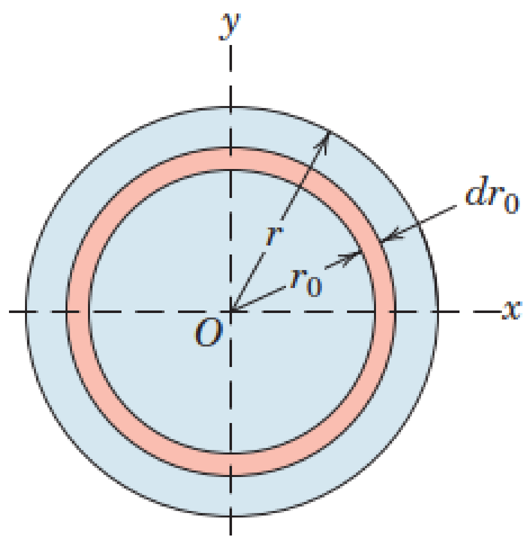
$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

A transfer from the centroidal axis to the *x′*-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4}$$

Sample Problem A/3

- Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.



Sample Problem A/3

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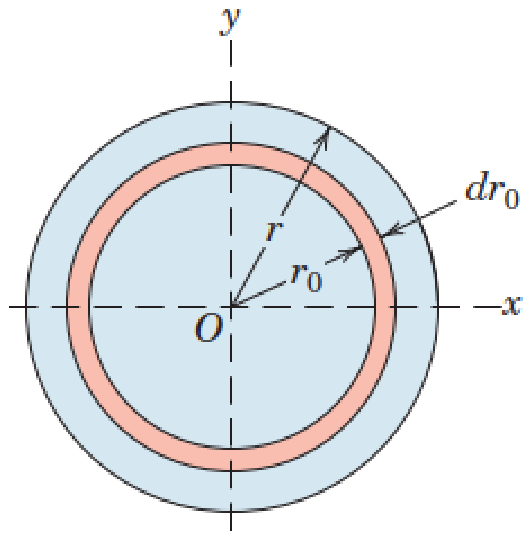


The elemental area is $dA = 2\pi r_0 dr_0$

$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2$$

The polar radius of gyration is

$$k = \sqrt{\frac{I}{A}} \qquad k_z = \frac{r}{\sqrt{2}}$$

By symmetry $I_x = I_y$

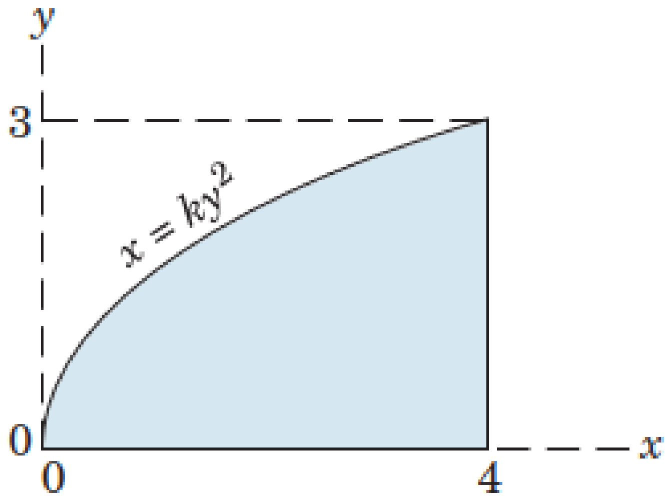
$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2}$$

Sample Problem A/4

- Determine the moment of inertia of the area under the parabola about the x-axis. Solve by using (a) a horizontal strip of area and (b) a vertical strip of area.



Sample Problem A/4

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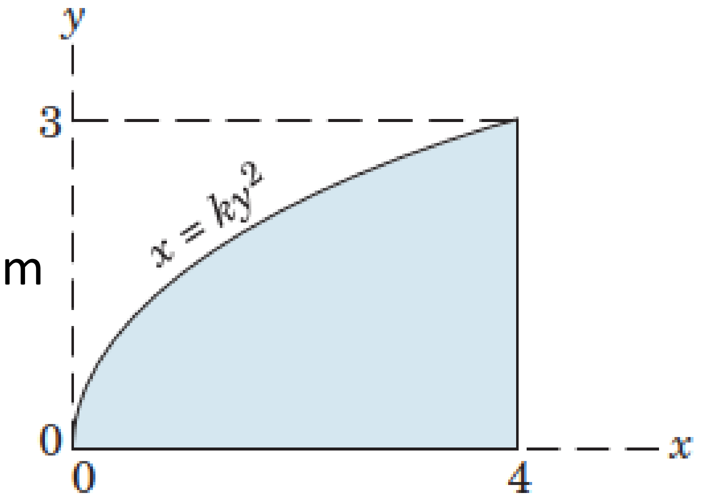


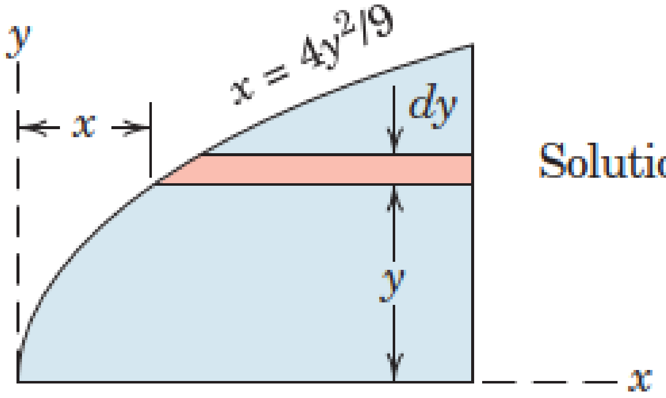
Horizontal strip: Since all parts of the horizontal strip are the same distance from the x-axis,

$$[I_x = \int y^2 \, dA]$$

where $dA = (4 - x)dy = 4(1 - y^2/9)dy$

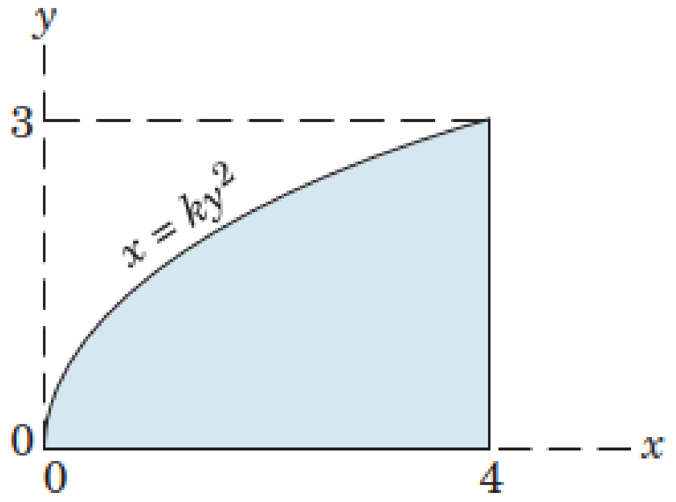
integrating

$$I_x = \int_0^3 4y^2\left(1 - \frac{y^2}{9}\right) dy = \frac{72}{5} = 14.4 \text{ (units)}^4$$



Sample Problem A/4

- Determine the moment of inertia of the area under the parabola about the x-axis. Solve by using (a) a horizontal strip of area and (b) a vertical strip of area.

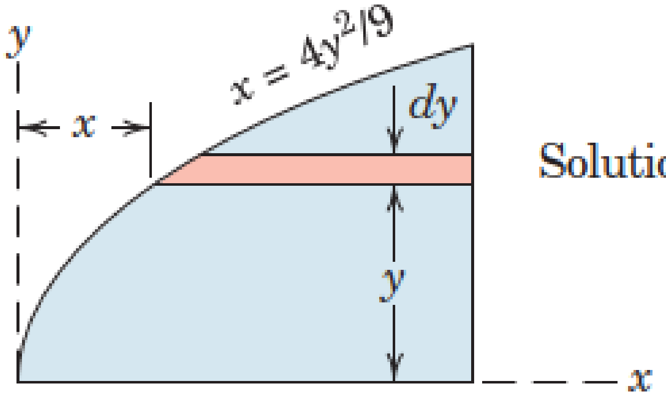


Vertical strip: Here all parts of the element are at different distances from the x-axis, so we must use the correct expressions for the moment of inertia of the elemental rectangle about its base is $bh^3/3$

$$dI_x = \frac{1}{3}(dx)y^3$$

$$I_x = \frac{1}{3}\int_0^4 \left(\frac{3\sqrt{x}}{2}\right)^3 dx = \frac{72}{5} = 14.4\ (units)^4$$

Where $y = 3\sqrt{x}/2$



Solution (a)

A/3 Composite Areas

- It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape.
- Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity.
- The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis.
- It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

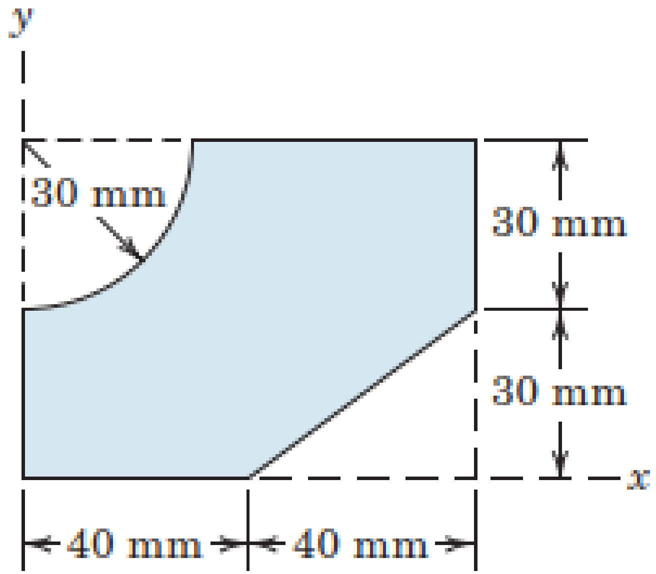
A/3 Composite Areas

- When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A, its centroidal moment of inertia $\bar{I}$, the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2.
- For any one of the parts the moment of inertia about the desired axis by the transfer-of-axis theorem is $\bar{I} + Ad^2$.
- Thus, for the entire section the desired moment of inertia becomes

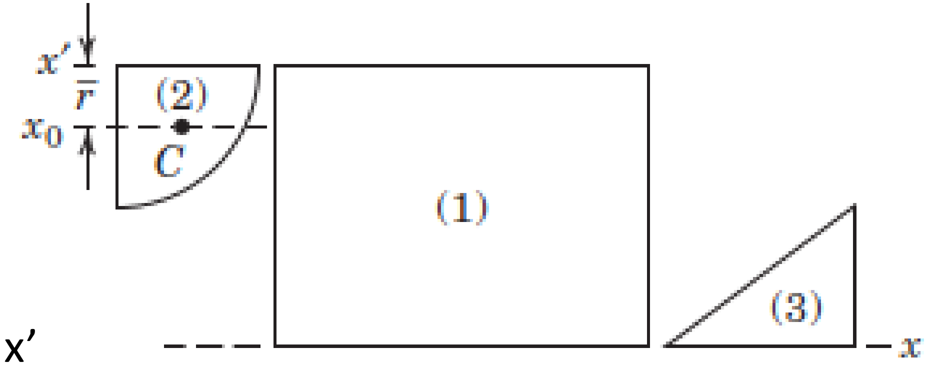
$$I = \Sigma\bar{I} + \Sigma Ad^2$$

SAMPLE PROBLEM A/8

Calculate the moment of inertia and radius of gyration about the x-axis for the shaded area shown. Wherever possible, make expedient use of tabulated moments of inertia.



SAMPLE PROBLEM A/7



For the rectangle the moment of inertia about the x-axis,

$$I_x = \frac{1}{3}Ah^2 = \frac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\text{ mm}^4$$

The moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\text{ mm}^4$$

We now transfer this result through the distance $\bar{r} = \frac{4r}{3\pi} = \frac{4(30)}{3\pi} = 12.73\ mm$ by transfer of axis theorem to get centroidal moment of inertia of part (2):

$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right]$$
$$= -0.0445(10^6)\text{ mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

$$[I = \bar{I} + Ad^2] \qquad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2$$
$$= -1.624(10^6)\text{ mm}^4$$

SAMPLE PROBLEM A/8

Finally, the moment of inertia of the negative triangular area (3) about its base,

$$I_x = -\frac{1}{12}bh^3 = -\frac{1}{12}(40)(30)^3 = -0.90(10^6)\ \text{mm}^4$$

The total moment of inertia about the x-axis of the composite area is, consequently,

$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4$$

The radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm}$$