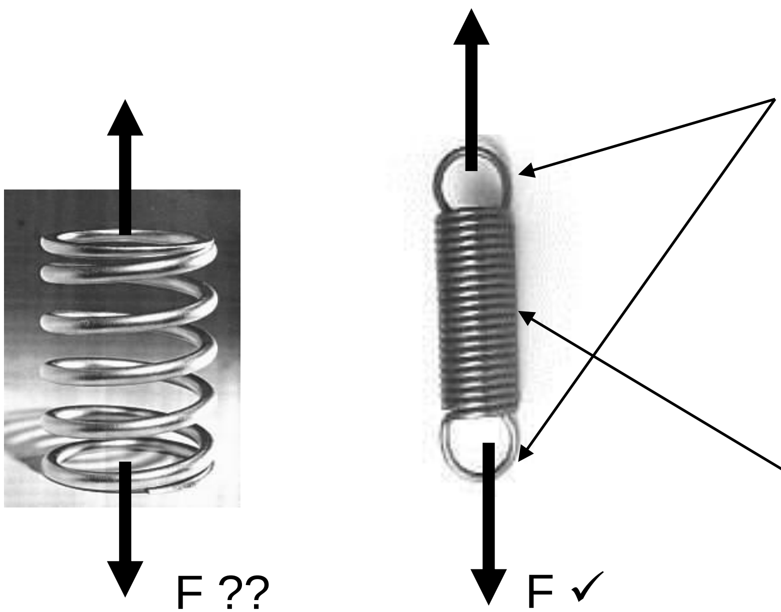
ME 308
MACHINE ELEMENTS II

CHAPTER 2

SPRING DESIGN
PART_2

2.8 EXTENSION SPRINGS

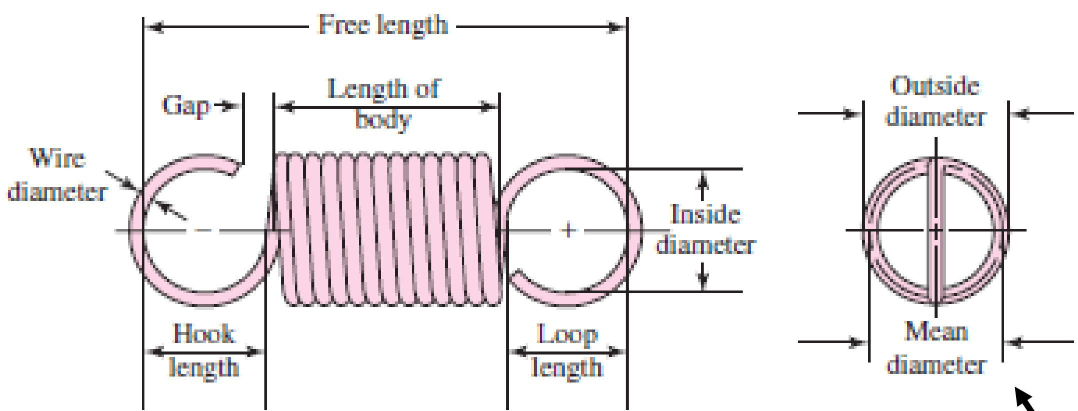
- Extension springs are designed to carry tensile loads since compression springs are not suitable to carry tensile loads.

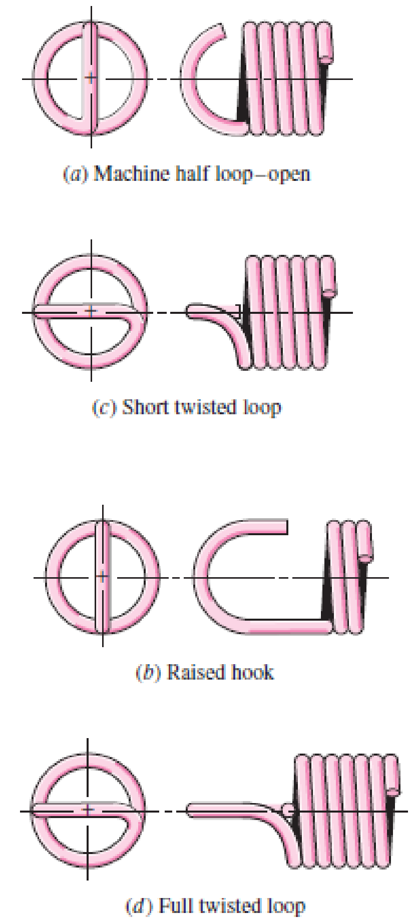


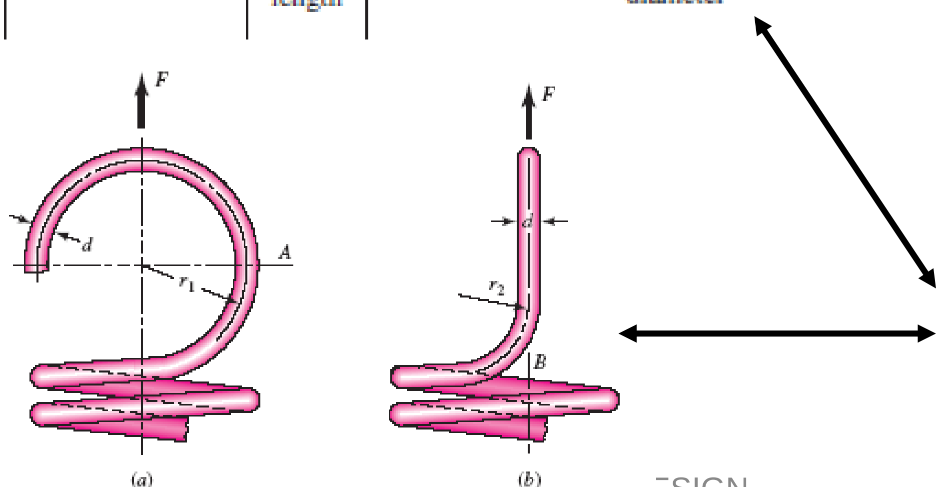
Hooks are used at two ends of the extension spring for the loads to be applied.

In addition, coils are no more seperated from each other, on the contrary coils are made to be in close contact with each other to resist tensile loads.

- There are different hook configurations of extension springs.
- However the most widely used one is the circular shape hook with hook diameter equal to coil diameter.
- <u>‘Hooks’ at two ends of the spring</u> are made by forming the ends of the wire

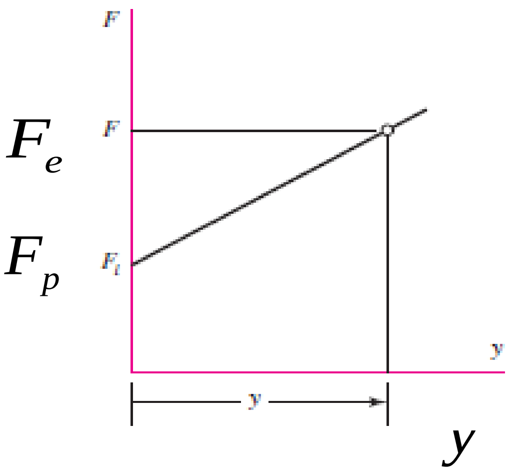




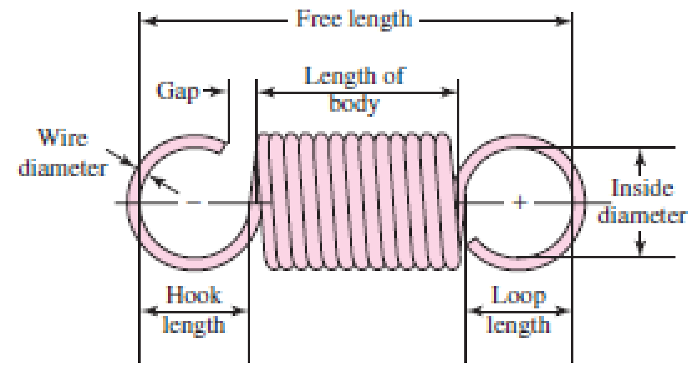


Since extension springs are produced as close-wound (shut coils) <u>there is always a preload (F_p) on the spring</u>.

This pre-load has to be overcome by the applied external load (F_e) to create an elongation on the spring.



$$k = \frac{F_e - F_p}{\Delta y} = \frac{d^4 G}{8 D^3 N_a}$$



$$L_b = body\ length = d\left(N_a + 1\right)$$

Due to preload (F_p) the coils are always under a pre-stress even if there is no external load on the spring

$$\tau_p = K_S \frac{8 F_p D}{\pi d^3} \quad \text{if } (F_e =< F_p)$$

$$\tau = K \frac{8 F_e D}{\pi d^3} \quad \text{if } (F_e > F_p)$$

The deflection δ_y in an extension spring under an external load of (F_e) is calculated as:

$$k = \frac{F_e - F_p}{\delta_y} = \frac{d^4 G}{8 D^3 N_a} \longrightarrow \delta_y = \frac{8 D^3 N_a \left(F_e - F_p\right)}{d^4 G}$$

where $$N_a = N_{body} + \frac{G}{E}$$ and used in the deflection formula to include effect of hook end deformations

The critical natural frequency of extension springs is again

$$f_n = \frac{1}{2}\sqrt{\frac{kg}{W_a}} \quad Hz\left(cycle / \sec\right)$$

$$W_a = \frac{\pi^2 d^2 D N_t \rho}{4}$$

ρ is the material weight density (*N/m*³)

and $f_n \geq 15 f_{force}$ *is still suggested for a reliable functioning.*

2.8.1 Extension Springs Under Load

Extension spring coil body is similar to compression spring coil body (except close wound condition).

Therefore similar shear stress (as in the case of comp. springs) occurs in the coil body of the extension spring under tensile load.

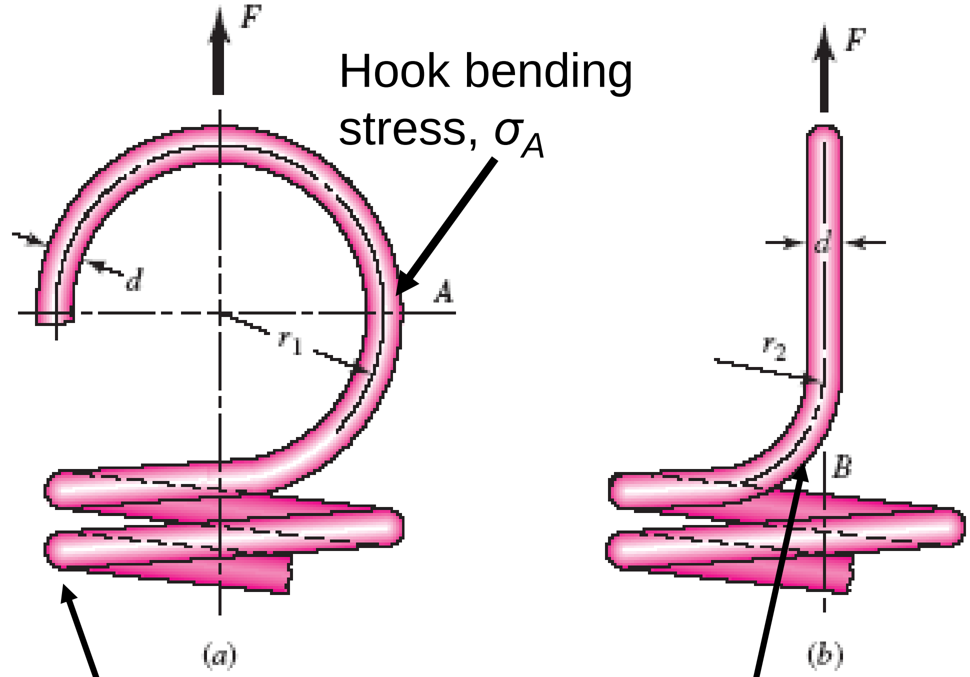
In addition to this, two more stresses occur in extension springs:

One; <u>torsional shear stress in the hook</u>

Two : <u>bending stress in the hook</u>.

Hence, extension springs should be designed (or checked) <u>against three different failures of the spring material (body shear, hook shear and hook bending)</u>

Hook bending stress, σ_A



Body coil shear stress τ (same as in compression springs)

hook shear stress, τ_B

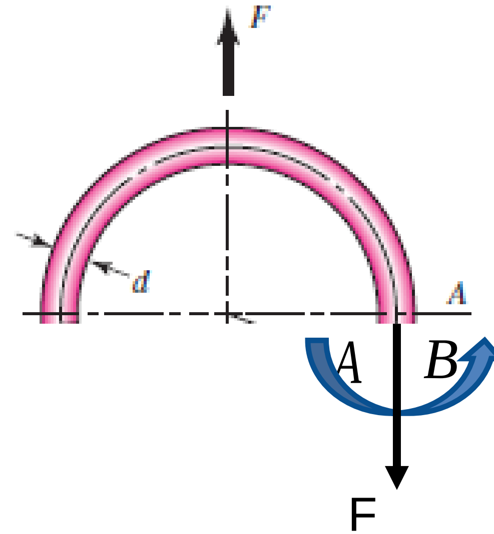
At point A

$$\sigma = \frac{F}{A} + \frac{Mc}{I}$$

but due to curvature effect

$$\sigma = \frac{F}{A} + K\frac{Mc}{I}$$

normal stress in the hook



$$M = Fr_m$$

$$\sigma_A = F\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right]$$

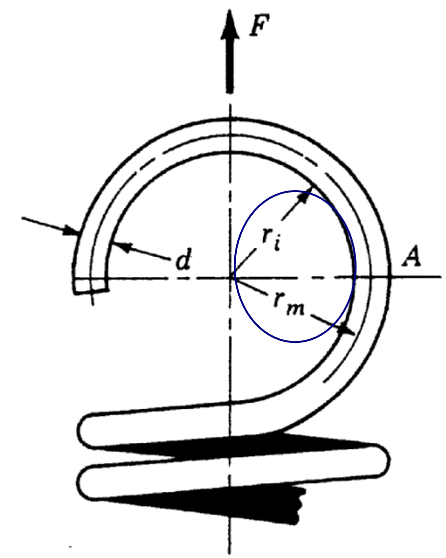
where

$$(K)_A = \frac{r_m}{r_i}$$

$$A = \pi \frac{d^2}{4}$$

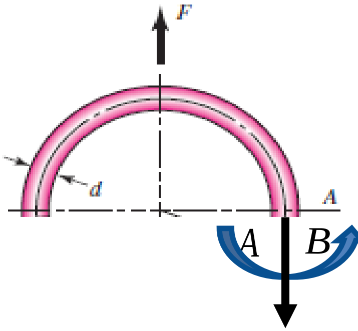
$$I = \pi \frac{d^4}{64}$$

$$c = \frac{d}{2}$$

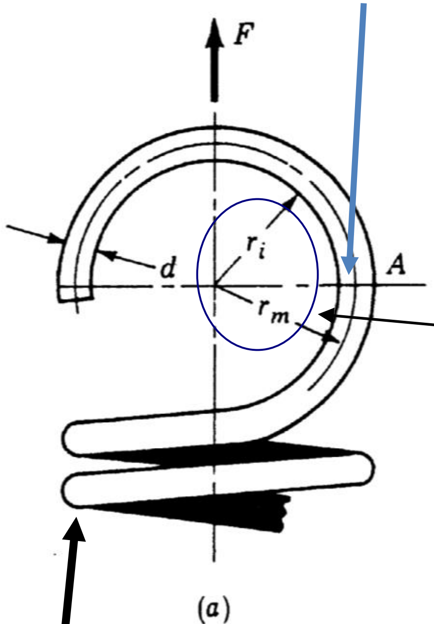


(a)

$$\sigma_A = F\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right]$$

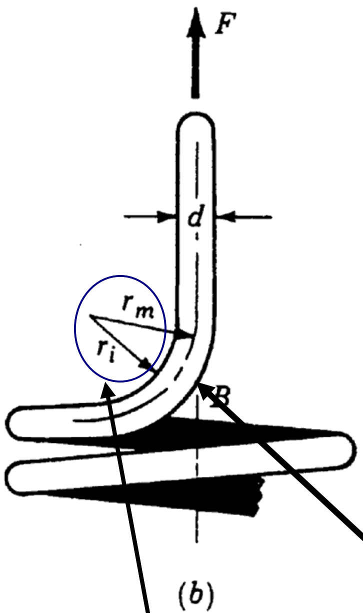


$$M = Fr_m$$



$$(K)_A = \frac{r_m}{r_i}$$

Bending plane



Torsion plane

$$\tau = K_S \frac{8F_e D}{\pi d^3}$$

$$\tau_B = (K)_B \frac{8FD}{\pi d^3}$$

$$(K)_B = \frac{r_m}{r_i}$$

Example 4: Extension Springs

A helical tension spring is made of 1.2 *mm* wire having yield strengths of 1280 *MPa* and 740 *MPa* in tension and torsion, respectively.

The springs has an OD of 12 *mm*, 36 active coils, and hook ends with mean radii at the ends 5.4 mm for bending and 3.0 *mm* for torsion.

The spring is pre-stressed to 75 *MPa* during winding which keeps it closed solid until an external load of sufficient magnitude is applied when wound, the distance between the hook ends is 70 *mm*.

a) What is the spring preload?

b) What load would cause yielding?

c) What is the spring rate?

d) What is the distance between the hook ends if the spring is extended until the stress just reach the yield strength.

SOLUTION:

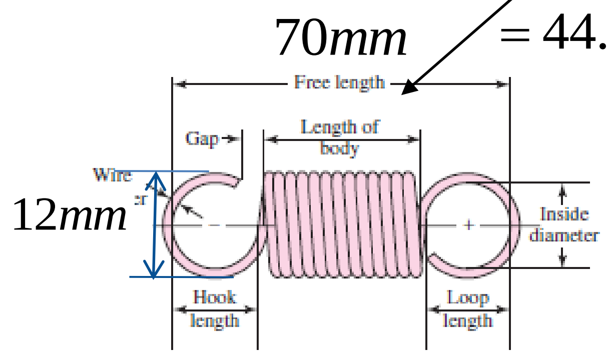
d=1.2 mm (4<C<12)

OD=12 mm ; D=OD-d =10.8 mm C=D/d=9 (4<9<12) OK

$$S_y = 1280\ MPa \quad S_{sy} = 740\ MPa \quad N_a = 36 \qquad K_s = 1 + \frac{0.5}{C} = 1.055$$

$$r_{m_b} = 5.4\ mm \rightarrow K_b = \frac{r_m}{r_i} = \frac{5.4}{5.4 - \frac{d}{2}} = 1.125$$

$$L_b = d\left(N_a + 1\right) = 44.4\ mm$$

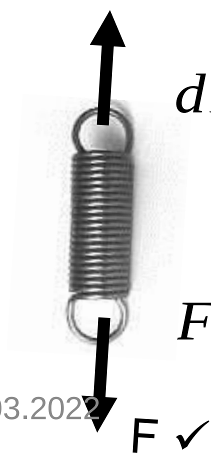


$$r_{m_t} = 3.0\ mm \rightarrow K_t = \frac{r_m}{r_i} = \frac{3}{3 - \frac{1.2}{2}} = 1.25$$

$$\tau_p = 75\ MPa, \& L_f = 70\ mm$$

a) since

$$\tau_p = 75\ MPa = K_s \frac{8FD}{\pi d^3} \rightarrow F_p = \frac{\tau_p \pi d^3}{K_s 8D}$$



$$d_w = 1.2\ mm$$

$$F_p = \frac{75\times10^6 \pi \left(1.2\times10^{-3}\right)^3}{1.055\times8\times10.8\times10^{-3}} = 4.467\ N$$

$$F_p = 450\ gr$$

$$\cong 0.45\ kg = 450\ gr$$

b) There are three types of yielding:

1) Yielding in hook due to bending effect $\rightarrow \sigma_h = K_b \dfrac{Mc}{I} + \dfrac{F}{A} = S_y$

2) Yielding in hook due to torsional shear $\rightarrow \tau_h = K_t \dfrac{8FD}{\pi d^3} = S_{sy}$

3) Yielding in coils due to torsional shear $\rightarrow \tau_{coil} = K_s \dfrac{8FD}{\pi d^3} = S_{sy}$

$$For\ 1 \rightarrow \sigma_h = 1.125 \frac{(Fr_{mb})\frac{d}{2}}{\frac{\pi d^4}{64}} + \frac{F}{\frac{\pi d^2}{4}} = 1280 \times 10^6\ Pa \rightarrow F_1 = 34.88\ N$$

$$For\ 2 \rightarrow \tau_h = 1.25 \frac{8 \times F \times 0.0108}{\pi (0.0012)^3} = 740 \times 10^6\ Pa \rightarrow F_2 = 37.19\ N$$

$$For\ 3 \rightarrow \tau_{coil} = 1.055 \frac{8 \times F \times 0.0108}{\pi (0.0012)^3} = 740 \times 10^6\ Pa \rightarrow F_3 = 44.07\ N$$

The smallest is the critical (safest one) ***F_{max} = 34.88 N*** cause yielding in the hook.

c) $$k = \frac{d^4 G}{8D^3 N_a} = \frac{(0.0012)^4 \times 79.3 \times 10^9}{8 \times (0.0108)^3 \times 36}$$

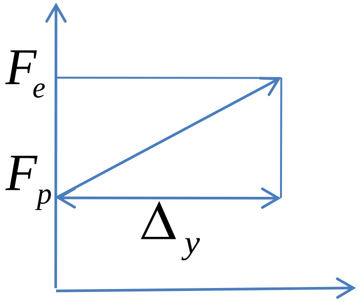
$$k = 453.2\ N/m$$

d) Since

$$F_{\max} = F_e = 34.88\ N$$

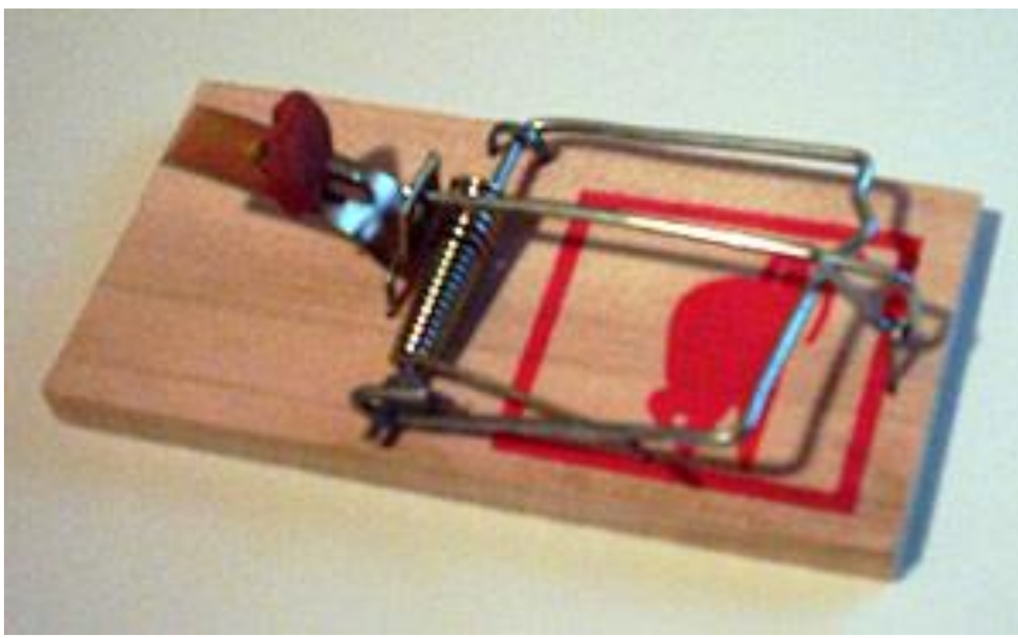
$$\delta_y = \frac{F_e - F_p}{k} = \frac{34.88 - 4.467}{453.2} = 0.0671\,m = 67.1\,mm$$

$$L_{new} = L_f + \delta_y = 70 + 67.1 = 137.1\,mm$$

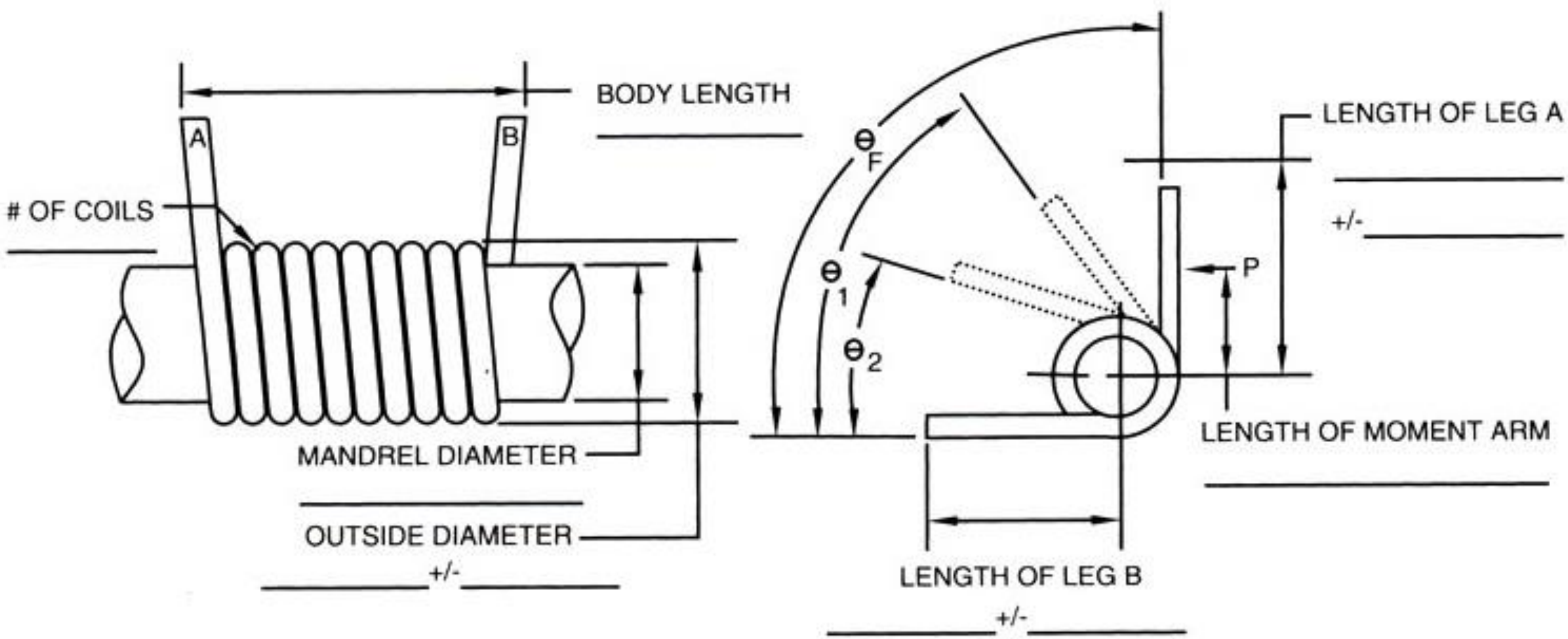


2.9 HELICAL TORSION SPRINGS

- Here are some examples of torsion springs



- Here is the geometry of torsion springs



And these are the different legs/ends of the torsion springs

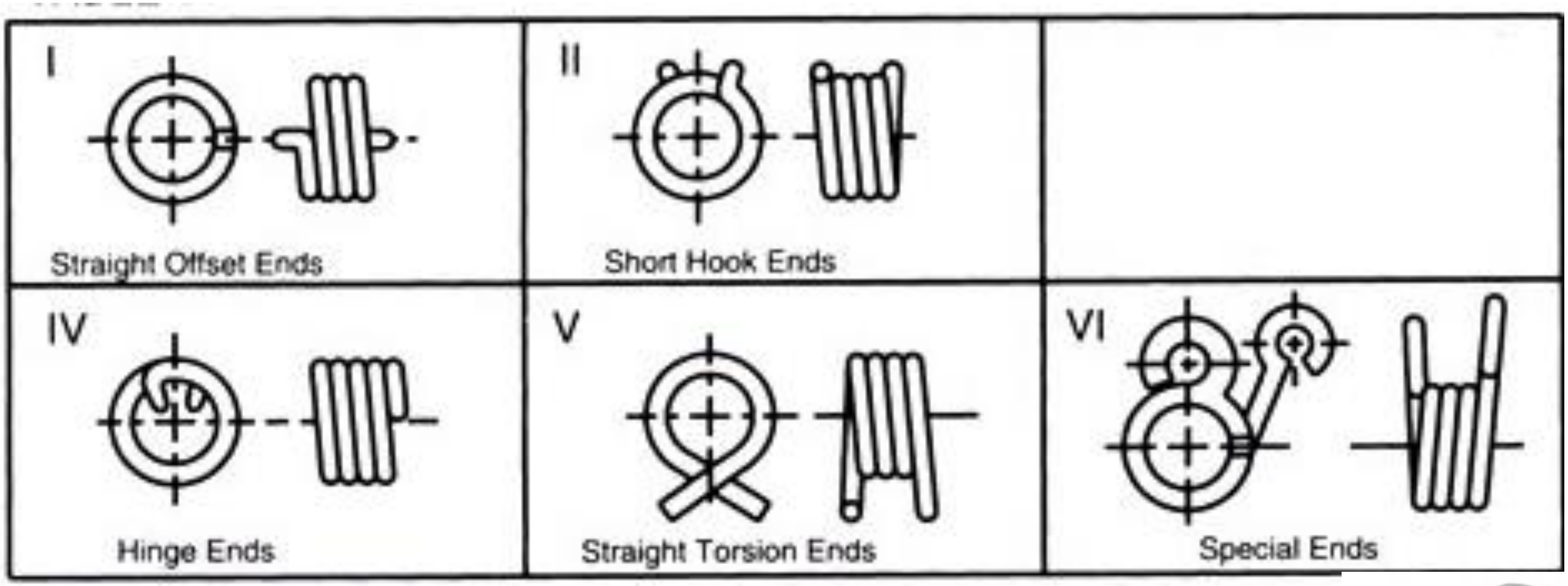
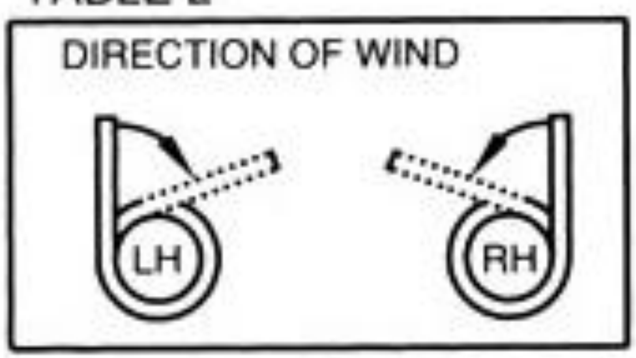
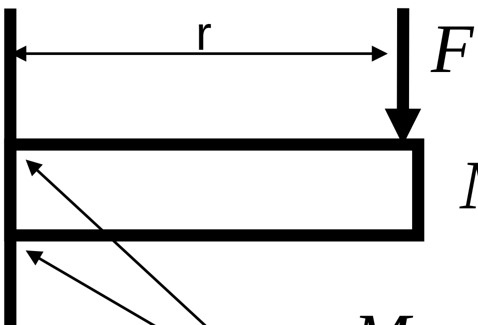


TABLE 2



2.9.1 Torsion Springs Under Load

While the torsion springs are used in applications where torque is required (e.g. door hinges, dress catcher etc.) the wire itself is subjected to a bending moment of *M= F*r* which produces a normal stress in the wire

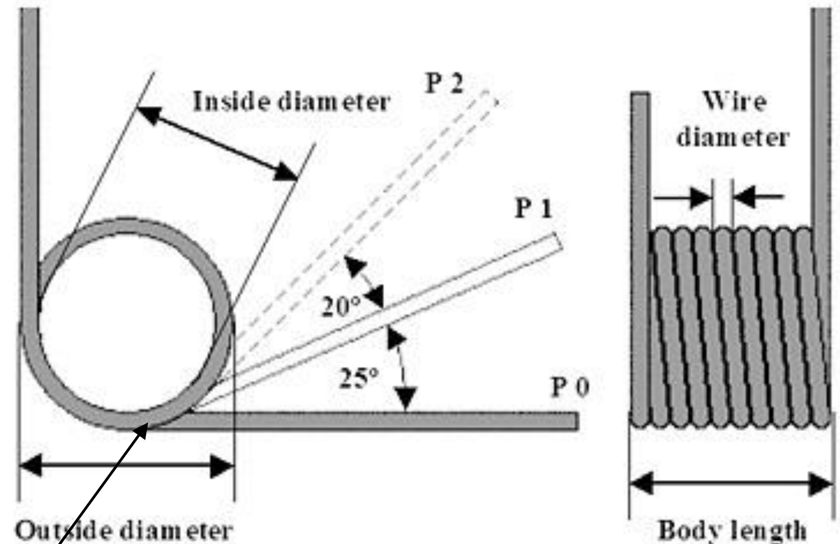


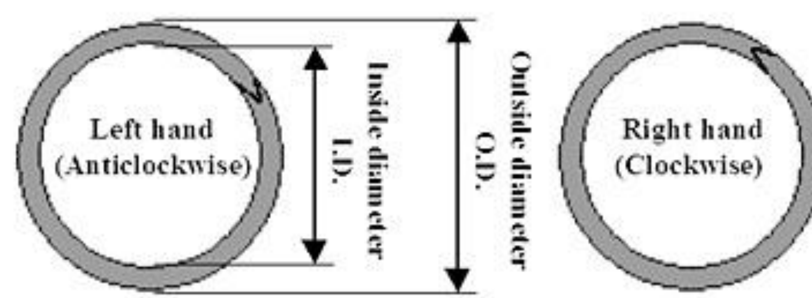
$$M = F \times r$$

$$\sigma = \frac{Mc}{I}$$

$$\sigma = \frac{Mc}{I} \times K$$

stress multiplier due to curvature of the coil-arm connection





In curved beams of torsion springs

$$C = \frac{D}{d}$$

$$\sigma_A = K_o \frac{Mc}{I} \quad \text{at outer fiber}$$

$$I = \frac{\pi d^4}{64}$$

$$\sigma_B = K_i \frac{Mc}{I} \quad \text{at inner fiber}$$

$$c = \frac{d}{2}$$

$$K_o = \frac{4C^2 + C - 1}{4C(C+1)} \quad \text{and} \quad K_i = \frac{4C^2 - C - 1}{4C(C-1)}$$

Whichever of the K_o *and* K_i is larger it is used in design calculations as *K.*

Thus,

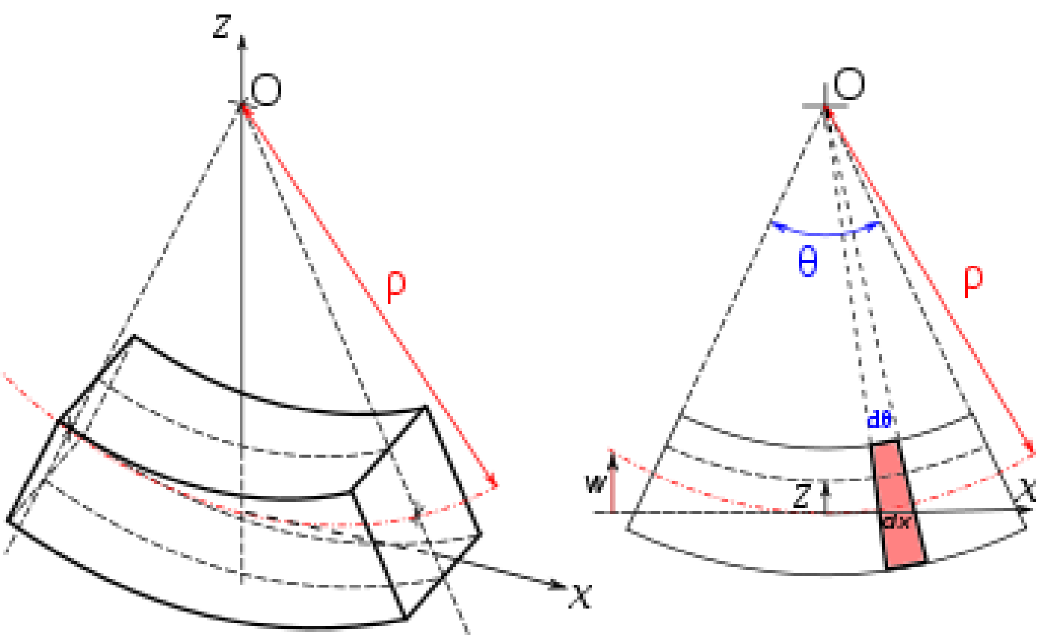
$$\sigma_{max} = K_{max} \frac{Mc}{I} = K_{max} \frac{32 F_{max} r}{\pi d^3}$$

this is the max. bending stress occurs in round-wire torsion springs

$$\sigma_{max} \le S_y / n \quad \text{for static design}$$

$$or \quad n_s = \frac{S_y}{\sigma_{max}} \ge 1.0$$

Deflection of torsion springs,



for an element of *dx* length

$$du = Strain\ energy = \frac{1}{2} M d\theta$$

Also

$$dx = \rho d\theta \quad or \quad \frac{1}{\rho} = \frac{d\theta}{dx}$$

and $$\frac{1}{\rho} = \frac{M}{EI} \left(from\ ME\ 223 \right)$$

Thus $$d\theta = \frac{M}{EI} dx$$

into $$du = \frac{1}{2} M d\theta = \frac{1}{2} M \frac{M dx}{EI}$$

$$du = \frac{M^2 dx}{2EI}$$

for small element dx; $$du = \frac{M^2 dx}{2EI}$$

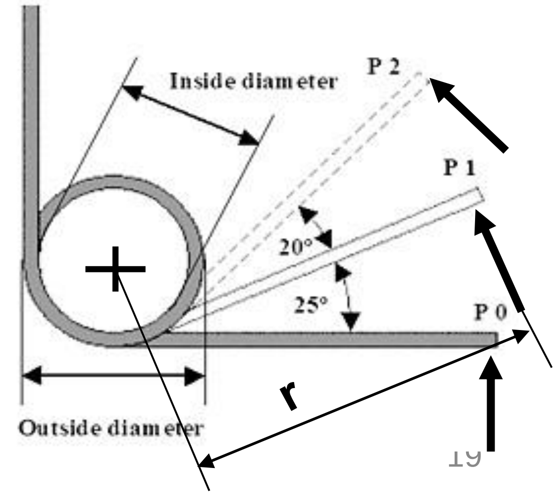
for whole length; $$u = \int_0^L \frac{M^2 dx}{2EI}$$

And for torsion springs; $$L = \pi D N_t \quad and \quad M = Fr$$

$$u = \int_0^{\pi D N_t} \frac{(Fr)^2 dx}{2EI} = \int_0^{\pi D N_t} \frac{F^2 r^2 dx}{2EI}$$

Using castiglione's theorem
for deflection: $$\frac{\partial u}{\partial F} = y$$

In torsion springs; $y = r\,\theta$



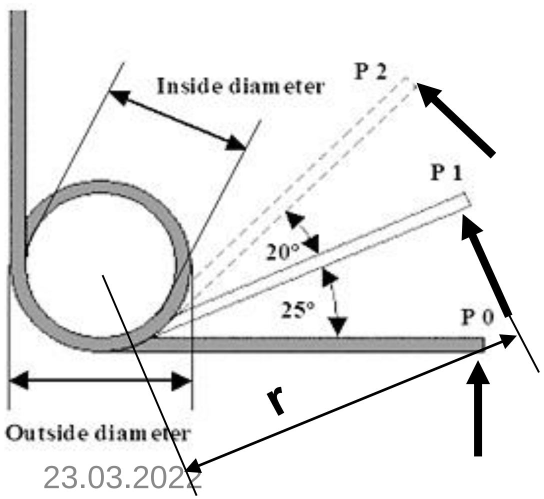
With relation $y = r\theta$

$$\frac{\partial u}{\partial F} = y$$

gives the deflection *y* along the direction of *F* under the effect of force *F*.

$$y = r\theta = \frac{\partial u}{\partial F} = \int_0^{\pi D N_a} \frac{2Fr^2 dx}{2EI} = \int_0^{\pi D N_a} \frac{Fr^2 dx}{EI} = \frac{Fr^2 x}{EI}\Bigg|_0^{\pi D N_a} = \frac{Fr^2 \pi D N_a}{EI} = r\theta$$

$$Thus \quad \theta = \frac{Fr\pi D N_a}{EI} \qquad where \quad I = \frac{\pi d^4}{64} \quad \theta = \frac{64 Fr D N_a}{d^4 E} \quad (rad)$$



this is the angular deflection of the spring in radians under the effect of force *F* at a distance of *r* from the spring center.

The torsional spring rate (similar to linear deflection $F=k\,x$) is then

$$k = \frac{d^4 G}{8D^3 N_a}$$

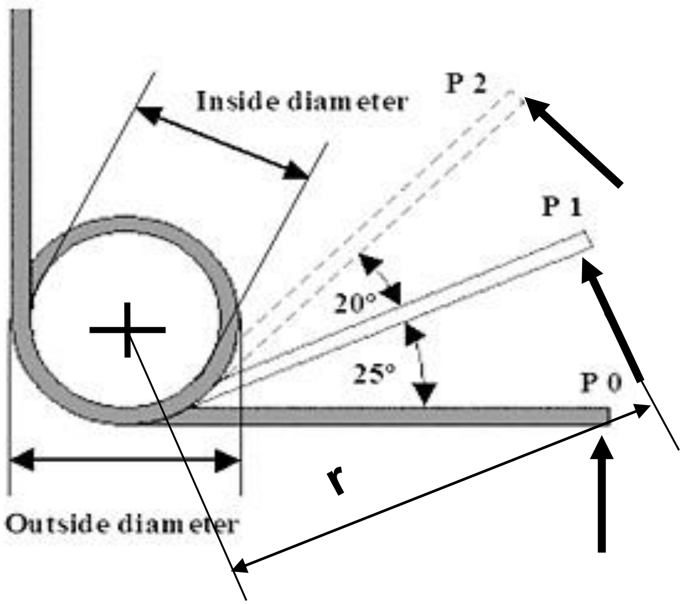
$$M = k\theta \quad or \quad k = \frac{M}{\theta} = \frac{Fr}{\dfrac{64FrDN_a}{d^4E}} = \frac{d^4E}{64DN_a} \quad in\ Nm/radians$$

$$\theta = \frac{64FrDN_a}{d^4E} \quad (rad)$$

$$or \quad k' = \frac{d^4E}{64DN_a} \quad \frac{Nm}{rad}\left(\times\frac{2\pi\ rad}{1\,rev}\right) = \frac{d^4E}{10.2DN_a} \quad in\ \frac{Nm}{rev}$$

$$or \quad k' = \frac{d^4E}{10.8DN_a} \quad \frac{Nm}{rev}$$

will give better results when compared with test results.

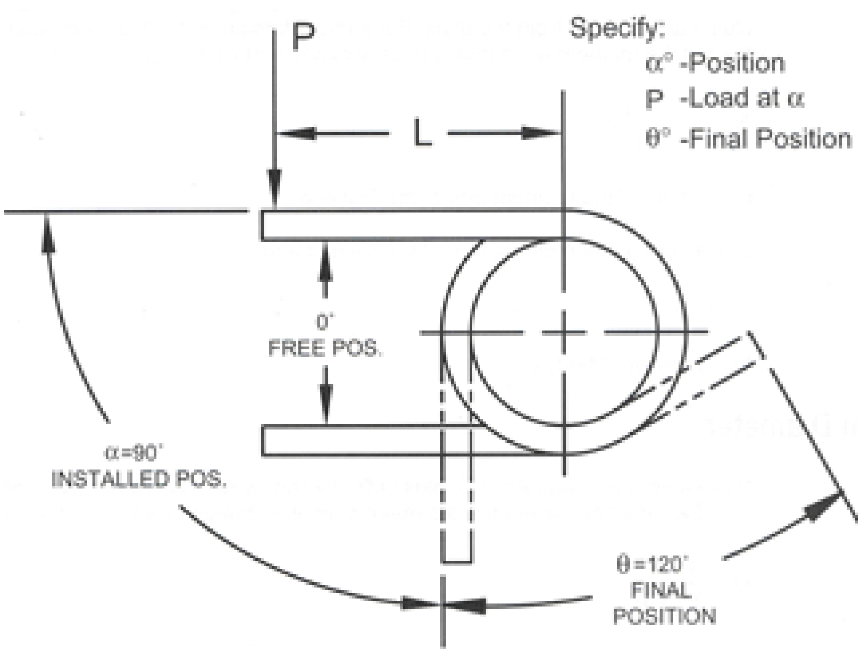


The compressive forces, *F's* winding up the spring, cause an inner diameter reduction in torsion spring.

Since the round bar (fitted inside the torsion spring) is rigid the inner diameter of the spring can't be less than diameter of the bar when the loads are applied.

$$D_i^{'} N_a^{'} = D_i N_a \qquad M = k^{'}\theta \quad or \quad \theta = \frac{M}{k^{'}} \quad (in\ Nm/revs)$$

$$Also \quad N^{'} = N_a + \Delta N \quad and \quad \Delta N = \frac{Fr}{k^{'}}$$



$where \quad D_i \quad and \quad N_a$

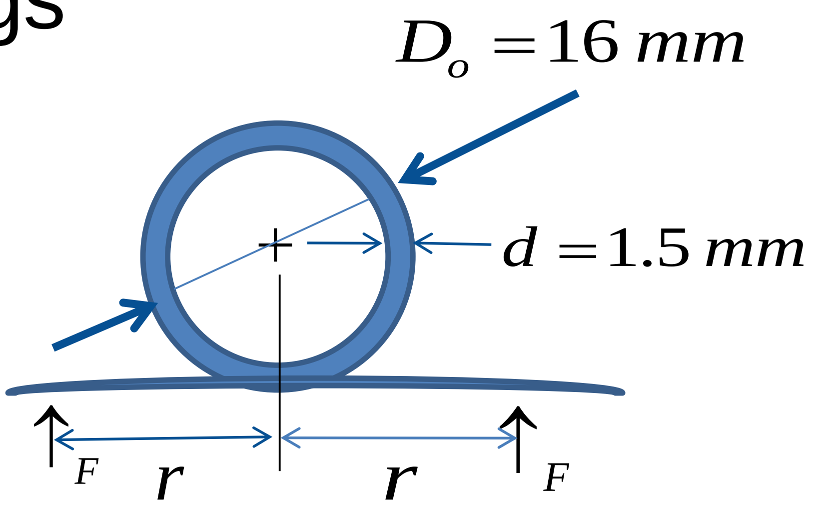
are the inside diameter and coil number when the load isn't applied.

$D_i^{'} \quad and \quad N^{'}$

are the inside diameter and coil number after the load is applied.

Example 5: for torsion springs

A stock torsion spring is made of 1.5 mm <u>music wire,</u> has 6 coils and straight ends 50 mm long and 180° apart. The outside diameter is 16 *mm*.



a) What value of torque *(Fxr)* would cause a maximum stress equal to the yield strength?

$$\sigma = K_{i,o}\frac{32Fr}{\pi d^3} \quad Where \quad Fr = ?$$

b) If the torque found in (a) is used as the maximum working torque, what will be the smallest value of the inside diameter. (D_i'=?)

$$D_i' = D_i N / N' = ?$$

c) Compute the angle of rotation corresponding to the torque found in (a)

$$\theta = \frac{Fr\,\pi D N_a}{EI} = ?$$

$$\theta = \frac{M}{k'} = in \quad revs$$

Answers : $\sigma = K_{i,o}\dfrac{32Fr}{\pi d^3}$ $\quad Where \quad Fr = ?$ $\qquad Fr = \dfrac{S_y \pi d^3}{32K_i}$

a)

$$\sigma = S_y \quad \rightarrow \quad S_{ut} = \frac{A}{d^m} = \frac{2170}{(1.5)^{0.146}} = 2045.27\ MPa$$

$$\sigma = S_y = 0.75 \times S_{ut} = 0.75 \times 2045.27 = 1534\ MPa$$

$$K_o = \frac{4C^2 + C - 1}{4C(C+1)} \ \&\ K_i = \frac{4C^2 - C - 1}{4C(C-1)} \qquad C = \frac{D}{d} = \frac{16 - 1.5}{1.5} = 9.67$$

$$K_i = \frac{4C^2 - C - 1}{4C(C-1)}$$

$$K_i = 1.08 > K_o = 0.9272$$

$$Fr_{\max} = \frac{S_y \pi d^3}{32K_i} = \frac{1534 \times 10^6 \times \pi (1.5 \times 10^{-3})^3}{32 \times 1.08}$$

$$Fr_{\max} = 0.47\ Nm = 470\ Nmm$$

b) $D_i^{'} N_a^{'} = D_i N_a$ $\quad D_i^{'} = D_i \dfrac{N_a}{N_a^{'}}$ $\quad M = k^{'}\theta \quad or \quad \theta = \dfrac{M}{k^{'}} \quad (in\ Nm/revs)$

$$D_i = D_o - 2d = 16 - 2 \times 1.5 = 13\,mm$$

$$N_a = 6$$

$$N_a^{'} = N_a + \Delta N = ?$$

$$\Delta N = \theta = n_{turn} = \frac{Fr}{k^{'}}$$

$$k^{'} = \frac{d^4 E}{10.8 D N_a} = \frac{\left(1.5 \times 10^{-3}\right)^4 \times 207 \times 10^9}{10.8 \times (0.0145) \times 6} = 1.115 \frac{Nm}{turn}$$

$$\Delta N = \frac{0.47}{1.115} = 0.421 \quad \text{turns} \longrightarrow D_i^{'} = 13 \frac{6}{6 + 0.421} = 12.147\,mm$$

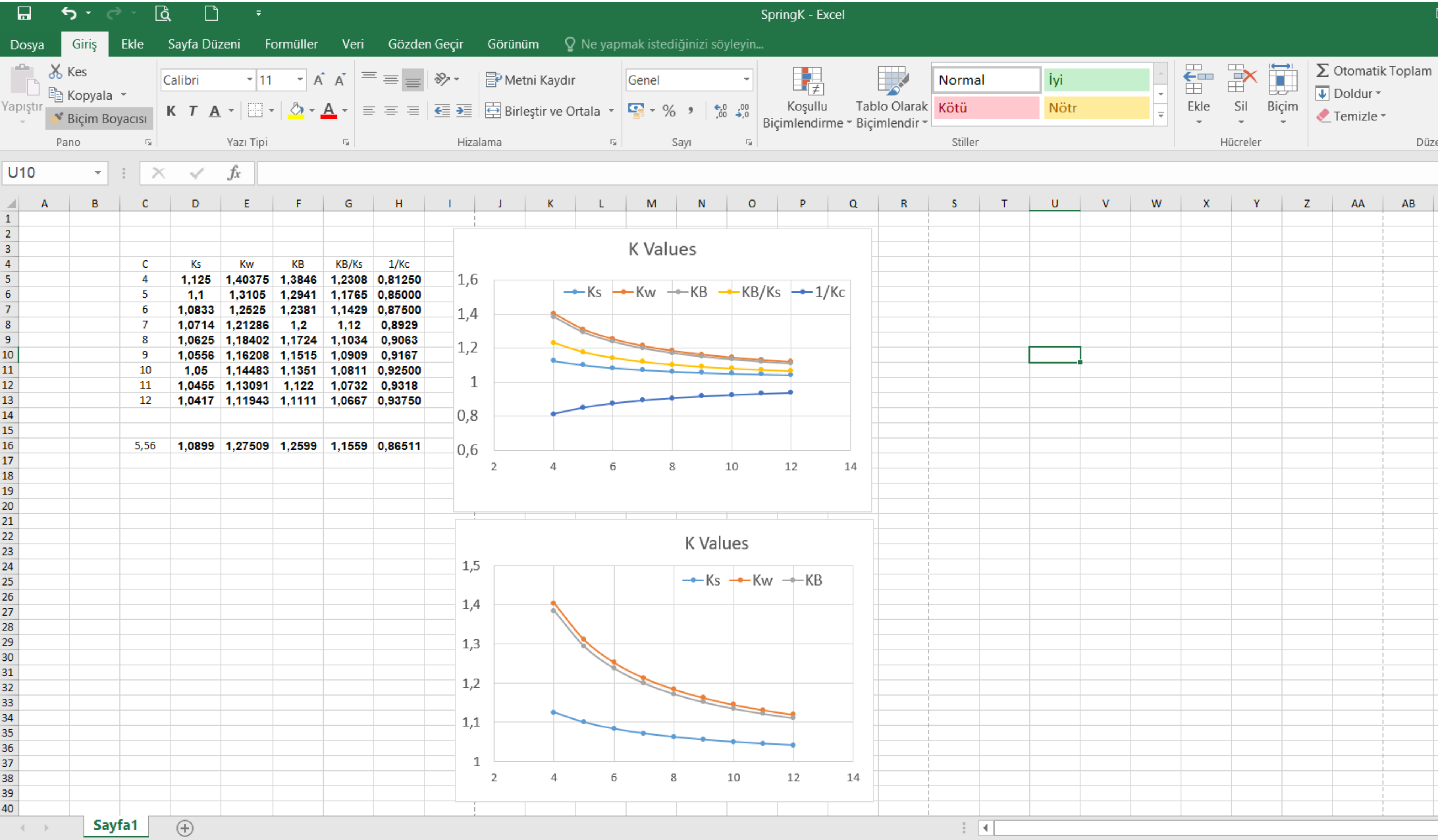
$$N_a^{'} = N_a + \Delta N = 6.0 + 0.421 = 6.421$$

c) $$\theta = n_{turn} \times 360 = 0.421 \times 360 = 151.56 \quad degrees$$

2.10 Design Procedure for the Helical Springs

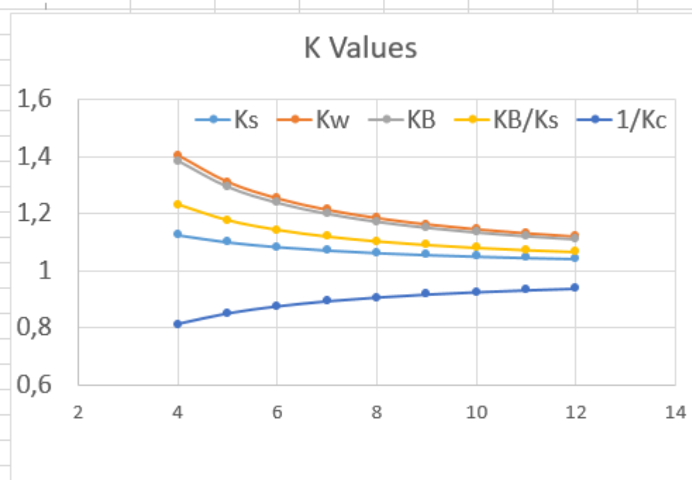
- Most of the spring parameters are unknown at the beginning of a design stage.
- Each design is an iteration procedure of assuming some parameters and values (material, *C, d, D or* D_o, D_i etc).
- Afterwards, you have to check whether other geometric constraints and failure criteria are satisfied or not. However, reaching a suitable solution may require lots of iteration and calculation steps
- To ease the design process, most spring design problems can be put into a tabulation-iteration form similar to (spreadsheets) as seen in following example:

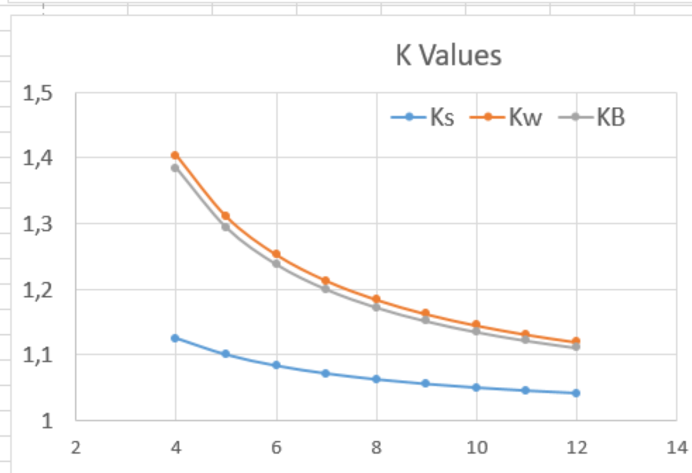
Assume d (mm)	**C**	K_s	τ_{max} **(MPa)**	S_{ut} **(MPa)**	S_y **(MPa)**	S_{sy} **(MPa)**	$\tau_{max} <? S_{sy}$ **Notes**
6							861>536 failure
7							549>521 failure
8							372<508 satisfactory



C	Ks	Kw	KB	KB/Ks	1/Kc
4	1,125	1,40375	1,3846	1,2308	0,81250
5	1,1	1,3105	1,2941	1,1765	0,85000
6	1,0833	1,2525	1,2381	1,1429	0,87500
7	1,0714	1,21286	1,2	1,12	0,8929
8	1,0625	1,18402	1,1724	1,1034	0,9063
9	1,0556	1,16208	1,1515	1,0909	0,9167
10	1,05	1,14483	1,1351	1,0811	0,92500
11	1,0455	1,13091	1,122	1,0732	0,9318
12	1,0417	1,11943	1,1111	1,0667	0,93750
5,56	1,0899	1,27509	1,2599	1,1559	0,86511

C	Ks	Kw	KB	KB/Ks	1/Kc
4	1,125	1,40375	1,3846	1,2308	0,81250
5	1,1	1,3105	1,2941	1,1765	0,85000
6	1,0833	1,2525	1,2381	1,1429	0,87500
7	1,0714	1,21286	1,2	1,12	0,8929
8	1,0625	1,18402	1,1724	1,1034	0,9063
9	1,0556	1,16208	1,1515	1,0909	0,9167
10	1,05	1,14483	1,1351	1,0811	0,92500
11	1,0455	1,13091	1,122	1,0732	0,9318
12	1,0417	1,11943	1,1111	1,0667	0,93750
5,56	1,0899	1,27509	1,2599	1,1559	0,86511





EXAMPLE 6:

A helical compression spring of hard-drawn wire with a mean diameter of 40 mm and squared and ground ends is assembled with a preload of 500 *N* and will operate to a maximum load of 1700 *N*.

a) Compute the wire diameter based on static failure with safety factor 1.25.

b) How many of total coils are required if the spring scale, *k* is required to be 127 *kN/m* (end condition squared and ground).

SOLUTION:

Table 2.1 Spring materials and constant for estimating tensile strength			
Material	**Size range (mm)**	**Exponent, *m***	**Constant, *A* (*MPa.mm^m*)**
Music wire	0.10-6.5	0.146	2170
Oil-tempered wire	0.50-12	0.186	1880
Hard-drawn wire	0.70-12	0.192	1750
Chrome-vanadium	0.80-12	0.167	2000
Chrome silicone	1.60-10	0.112	2000

a) For static failure $\tau_{\max} \leq S_{sy}$ $\tau_{\max} = K_s \dfrac{8F_{\max} D}{\pi d^3}$ and

Similarly $S_{sy} = 0.577 S_y$

$S_y = 0.75 S_{ut}$

$K_s = 1 + \dfrac{0.5}{C}$ *and* $C = \dfrac{D}{d}$

$S_{ut} = \dfrac{A}{d^m}$

Everything is dependent on d

Here again strength is dependent on d which is already unknown,

However we can make use of information in Table 2.1 for "Hard drawn wire" d = 0.70-12 mm

Here $\tau_{\max} \leq S_{sy}$

Given: D= 40 mm, squared and ground ends

$$\tau_{\max} = K_s \frac{8F_{\max} D}{\pi d^3} \quad \text{where} \quad K_s = 1 + \frac{0.5}{C} \quad and \quad C = \frac{D}{d}$$

$$S_{sy} = 0.577 S_y \quad \text{and} \quad S_y = 0.75 S_{ut} \quad \text{and} \quad S_{ut} = \frac{A}{d^m}$$

Thus if we use a table of iteration with *d* assumed (to be between *0.70 mm* and *12 mm*) and rest checked, we can reach a feasible solution

d (mm)	**C**	K_s	$\tau_{\max}$ **MPa**	S_{ut} **MPa**	S_y **MPa**	S_{sy} **MPa**	$n_s = S_{sy}/\tau_{\max}$ **Notes**
6	6.67	1.075	861	1240.6	930.3	536.8	861>536 failure
7	5.71	1.0875	549	1204.4	903.3	521.2	549>521 failure
8	5.0	1.10	372	1174	880.5	508.0	372<508 satisfactory

$$n_s = 508/372 = 1.36 \quad \textit{Satisfactory}$$

b)

$$N_t = N_a + N_e$$ For squared and ground ends → $N_e = 2$

$$N_a = ?$$

$$k = \frac{d^4 G}{8 D^3 N_a}$$

$$N_a = \frac{d^4 G}{8 D^3 k} = \frac{(0.008)^4 \times 79.3 \times 10^9}{8 \times (0.04)^3 \times 127 \times 10^3}$$

$$N_a = 5$$

$$N_t = 5 + 2 = 7 \; coils$$

2.11 OPTIMIZATION OF SPRING DESIGN

Springs are usually optimized in two categories:

1)Objective is <u>to minimize</u>

a) weight

b) volume

c) wire diameter

d) Length

e) spring rate

2)Objective is <u>to maximize</u>

a) work done by spring *(W=F*y)*

b) Deflection

c) factor of safety

d) Reliability

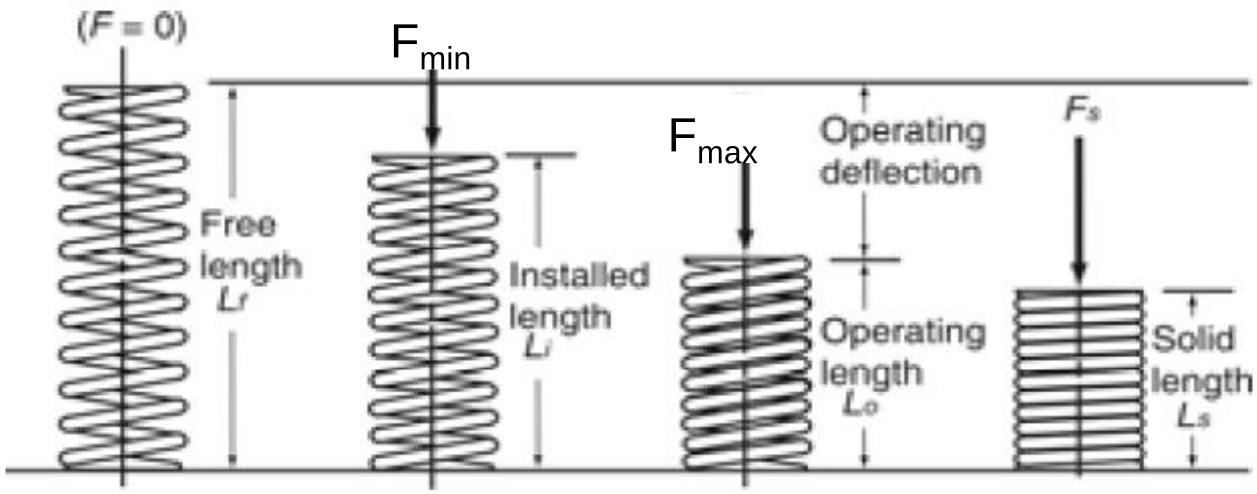
e) fatigue strength

In both categories of the optimization all the design requirements have to be satisfied e.g.

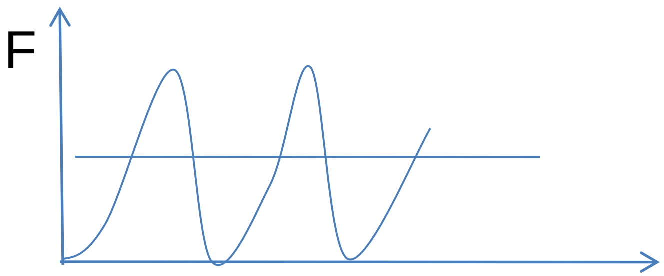
- Static safety factor,
- buckling,
- critical frequency,
- fatigue safety factor,
- geometrical constraint (*OD, ID* etc.)

2.12 Fatigue Loading of Springs

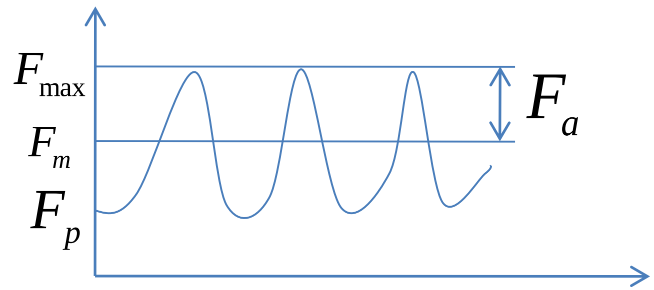
- In most applications springs are subjected to fatigue loading since they have to deflect between some points.
- The life of the springs may change from a few thousands cycle to millions of cycle (as in the valve spring application of automotive vehicles)
- Contrary to the rotating shafts under a vertical force in which completely reversed stresses are quite ordinary, springs can only be used either as compression or as tension but not together and most of the time they are installed with a preload.



Thus the stress-time diagram of



or



occur in helical springs.

The worst condition is with no-preload ($\tau_{min} = 0$)

In designing springs to resist fatigue failure, we start with calculating <u>alternating and mean components of the force</u>.

$$F_a = \frac{F_{max} - F_{min}}{2} \quad \text{and} \quad F_m = \frac{F_{max} + F_{min}}{2}$$

where

$$K_s = 1 + \frac{0.5}{C}$$

$$\tau_a = K_s \frac{8F_a D}{\pi d^3} \quad \text{and} \quad \tau_m = K_s \frac{8F_m D}{\pi d^3}$$

$$C = \frac{D}{d}$$

- Springs under varying (fatigue) loads should always be checked against both static failure and fatigue failure.
- If the spring is compression type this is done for the shear stress in the body of the spring
- If the spring is extension type then we have to check all three conditions; shear in body, shear in hook and bending in hook.
- If the spring is torsion type then we have to check bending in the arm/leg of the spring

For example (Chapter 7, Eq.s 7.37 and 7. 38)

1)Check for static safety $S_{sy} \geq \tau_m + \tau_a = \tau_{\max} \times n$

2)Check for fatigue safety $\tau_a \leq S_{se}/n$ for infinite life

For example

1)Check for static safety $$S_{sy} \geq \tau_m + \tau_a = \tau_{max} \times n$$

2)Check for fatigue safety $$\tau_a \leq S_{se}/n$$ for infinite life

An extended study(11) of available literature regarding torsional fatigue found that for polished, notch-free, cylindrical specimens subjected to torsional shear stress, the maximum alternating stress that may be imposed without causing failure is *constant* and independent of the mean stress in the cycle provided that the maximum stress range does not equal or exceed the torsional yield strength of the metal. With notches and abrupt section changes this consistency is not found. Springs are free of notches and surfaces are often very smooth. This failure criterion is known as the *Sines failure criterion* in torsional fatigue.

10 F. P. Zimmerli, "Human Failures in Spring Applications," *The Mainspring,* no. 17, Associated Spring Corporation, Bristol, Conn., August–September 1957.
11Oscar J. Horger (ed.), *Metals Engineering: Design Handbook,* McGraw-Hill, New York, 1953, p. 84.

For fatigue safety $\tau_a \leq S_{se}/n$ should be satisfied for infinite life

$$S_{se} = k_c \times k_d \times k_e \times S_{se}^{'}$$

$S_{se} =$ endurence limit strength in shear with relebility life, temperature and stress concentration factor.

where

$k_c \rightarrow reliability\ factor$

$k_d \rightarrow temperature\ factor$

$$k_e = \frac{1}{K_c} \rightarrow stress\ concentration\ factor$$

$$K_c = \frac{K}{K_s} = \frac{wahl\ correction\ factor}{stress\ multiplier} = \frac{\frac{4C-1}{4C-4} + \frac{0.615}{C}}{1 + \frac{0.5}{C}}$$

$$S_{se}^{'} = 310\ MPa\ (45\ kpsi)\ for\ unpeened\ springs$$

$$S_{se}^{'} = 465\ MPa\ (67.5\ kpsi)\ for\ peened\ springs$$

(corrected for k_a and k_b)

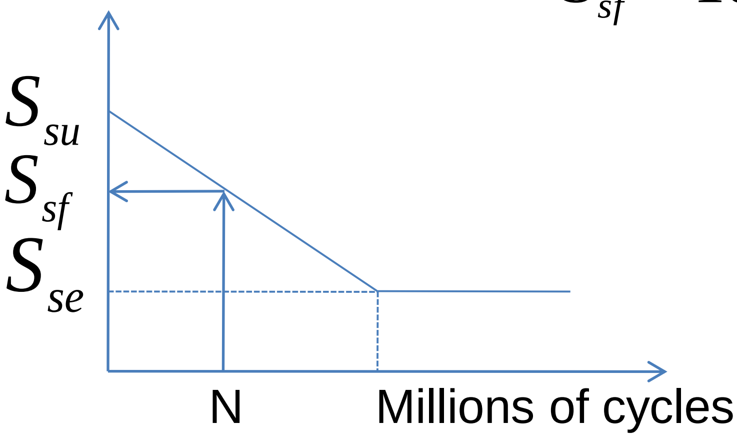
For finite life cases where $\left(\tau_a > S_{se} \quad ; \quad and \quad \tau_a = S_{sf}\right)$

Relation $S_{sf} = 10^c \times N^b$ is used where

$$c = \log \frac{\left(0.8 \times S_{su}\right)^2}{S_{se}}$$

$$b = -\frac{1}{3} \log \frac{0.8 \times S_{su}}{S_{se}}$$

$$S_{su} = 0.60 \times S_{ut}$$



S_{su} , The ultimate shear strength, Or torsional modulus of rupture

$$S_{ut} = \frac{A}{d^m} \quad or \quad S_{ut} = 3.45\ HB\ in\ MPa$$

$$S_{ut} = 500\ HB\ in\ psi$$

When analyzing or designing springs to resist fatigue, it is always important to <u>check critical (natural) frequency</u> to be sure that spring surge will not be a problem;

The critical natural frequency of torsion springs is again

$$f_n = \frac{1}{2}\sqrt{\frac{kg}{W_a}} \quad Hz\left(cycle/\sec\right) \qquad \left(f_n = \frac{1}{2}\sqrt{\frac{k}{m}}\right)$$

$$W_a = A \times L \times \rho = \frac{\pi d^2}{4} \times \pi D N_t \times \rho = \frac{\pi^2 d^2 D N_t \rho}{4}$$

ρ is the material weight density (*N/m*^{3})

$$f_{force} \leq \frac{f_n}{15}$$

and $f_n \geq 15\, f_{force}$ *is still suggested for a reliable functioning.*

Example 7: Failure of Compression Springs

A compression coil spring with; d=12 mm, D_i = 140 mm, N_t = 12 coils, squared and ground ends, HB of 380 after heat treatment, L_f= 500 mm is assembled into a machine by compressing it to a length of 458 mm.

When the machine runs, the spring is compressed an additional 254 mm so that the maximum load on the spring corresponds to a spring length of 458-254= 204 mm and the minimum load to a length of 458 mm.

a) Calculate the spring rate (k).
b) Would this spring develop a permanent set if compresses solid? why?
c) Is the spring likely to buckle?
d) Based on 50 % reliability and infinite life, will the spring fail by fatigue?
e) What will be the safe fatigue life for the spring if it is to work with in an environment of 400 ºC with a reliability of 99 %?
f) What should be the maximum forcing frequency of the spring to prevent spring surge if spring is held between parallel flat plates?

SOLUTION:

a) $d = 12\ mm, \quad G = 79.3 \times 10^9\ Pa\ \left(N/m^2\right) \quad for\ steel$

$D = D_i + d = 140 + 12 = 152\ mm$

$N_a = N_t - N_e = 12 - 2 = 10\ coils$

$$k = \frac{d^4 G}{8 D^3 N_a}$$

$$k = \frac{(12)^4 \times 79.3 \times 10^3}{8 \times (152)^3 \times 10} = \quad \underline{\underline{5.85\ N/mm}} \quad or \quad \underline{\underline{5850\ N/m}}$$

b) In case of compressing solid, permanent set occurs if $\tau_s > S_{sy}$

$$S_{sy} = 0.577 \times 0.75 \times S_{ut} \; ; \quad and \quad S_{ut} = \frac{A}{d^m} \quad or \quad S_{ut} = 3.45\, HB\, MPa$$

$$S_{ut} = 3.45 \times 380 = \underline{\underline{1311\, MPa}}$$

$$S_{sy} = 0.577 \times 0.75 \times 1311 = 567\ MPa$$

$$K_s = ? \quad 1 + \frac{0.5}{C}$$

$$F_s = ? = k\left(L_f - L_s\right)$$

$$C = \frac{D}{d} = \frac{152}{12} = 12.66$$

$$4 \le C \le 12 \quad Acceptable$$

$$\tau_s = K_s \frac{8F_s D}{\pi d^3}$$

$$\tau_s = \underline{\underline{485\ MPa}}$$

$$F_s = 5.85\left(500 - 144\right)$$

$$F_s = \underline{\underline{2082.6\ N}}$$

$$L_s = N_t \times d$$

$$L_s = 12 \times 12 = 144\, mm$$

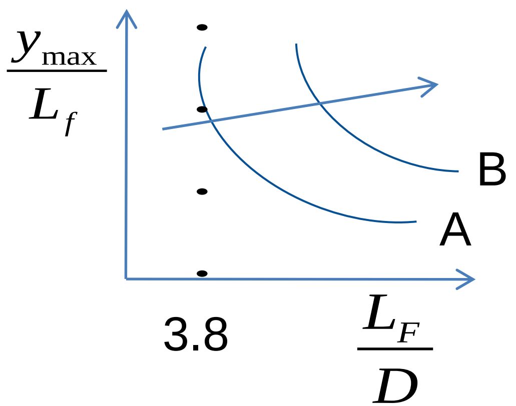
$$K_s = 1 + \frac{0.5}{12.66} = \underline{\underline{1.04}}$$

$$Since \quad \tau_s < S_{sy} \; \left(485 < 567\right)$$

<u>No-permanent set occurs when compressed solid</u>

c) Buckling ?

$$\frac{L_f}{D} = \frac{500}{152} = 3.3 < 3.8$$



there is <u>no risk of spring buckling</u> for either of the end conditions (A and B).

curve A: flat-to-rounded- end

B: flat-to-flat ends.

d) For infinite life condition; $\tau_a \leq S_{se} \quad or \quad n = \frac{S_{se}}{\tau_a} \geq 1.0$

$$if \quad \tau_a > S_{se} \quad or \quad n = \frac{S_{se}}{\tau_a} < 1.0$$

this means the spring has a <u>finite life</u> and <u>it fails based on infinite life requirements</u>

$$S_{se} = k_c \times k_d \times k_e \times S'_{se}$$

$$S'_{se} = 310\ MPa \qquad for\ unpeened\ springs$$

$$k_c = 1\left(Table\ 7.7 \qquad 50\%\ reliability\right)$$

$$k_d = 1 \quad No\ temp.\ is\ given$$

$$S_{se} = 1.0 \times 1.0 \times 0.934 \times 310$$

$$S_{se} = \underline{\underline{289.5\ MPa}}$$

$$k_e = \frac{1}{K_c}; \quad K_c = \frac{K_w}{K_s} = \frac{\frac{4C-1}{4C-4} + \frac{0.615}{C}}{1 + \frac{0.5}{C}} = \frac{1.113}{1.04}$$

$$k_e = 0.934$$

$$\tau_a = K_s \frac{8F_a D}{\pi d^3}$$

$$\tau_a = \underline{\underline{173\ MPa}}$$

$$F_a = \frac{F_{max} - F_{min}}{2}$$

$$F_a = \frac{k(500-204) - k(500-458)}{2}$$

$$F_a = k\frac{254}{2} = \underline{\underline{743\ N}}$$

$$n = \frac{S_{se}}{\tau_a} = \frac{289.5}{173} = 1.673 > 1.0$$

$$k = \underline{\underline{5.85\ N/mm}}$$

no fatigue failure based on infinite life.

spring has INFINITE LIFE.

e)

$$S_{se} = k_c \times k_d \times k_e \times S'_{se}$$

$$R = 99 \quad \% \rightarrow k_c = 0.814 \,(Table\ 7.7)$$

$$T = 400\ ^oC \rightarrow k_d = 0.5 \quad (Chpter\ 7, Eq(7.26))$$

$$k_e = \frac{1}{K_c} = 0.934 \qquad (Fig.7.12)$$

$$S_{se} = 0.814 \times 0.5 \times 0.934 \times 310$$

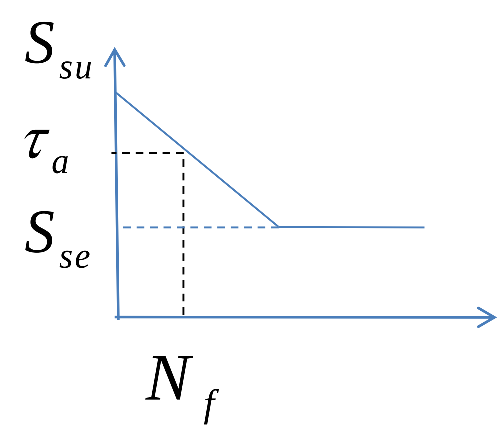
$$S_{se} = \underline{\underline{118\ MPa}}$$

$S_{se} =$ endurance limit strength in shear with reliability life, temperature and stress concentration factor.

$$\tau_a = 173\ MPa \quad as\ before$$

$$n = \frac{S_{se}}{\tau_a} = \frac{118}{173} = 0.68 < 1.0$$

<u>This means spring will have a FINITE life</u>; N_f



e)

For finite life region $\tau_a = S_{sf} = 10^c N_f^b$

$$\tau_a = 173\ MPa$$

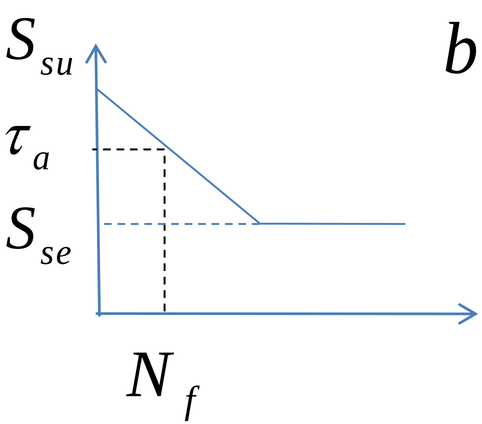
$$S_{ut} = 3.45\ HB\ MPa$$

$$S_{ut} = 3.45 \times 380 = 1311\ MPa$$

$$c = \log \frac{(0.8 \times S_{su})^2}{S_{se}}$$

$$S_{su} = 0.60 \times S_{ut}$$

$$S_{su} = 787\ MPa$$



$$b = -\frac{1}{3} \log \frac{0.8 \times S_{su}}{S_{se}}$$

$$c = 3.53$$

$$b = -0.243$$

$$173\ MPa = 10^{3.53} \times N_f^{-0.243}$$

$$N_f^{-0.243} = 0.051$$

$$N_f = 208300 \quad cycles$$

This is the maximum <u>safe number of cycles</u> that the spring can be loaded before failure.

f)

$$f_{force} \le \frac{f_n}{15} Hz\,(cycle/\sec)$$

$$f_n = \frac{1}{2}\sqrt{\frac{k(N/m)}{m_s(kg)}}$$

$$m_s = A \times L \times \rho = \frac{\pi(12)^2}{4} \times (\pi \times 152 \times 12) \times \left(10^{-9}\,\frac{m^3}{mm^3}\right) \times 7800\,\frac{kg}{m^3}$$

$$m_s = \underline{\underline{5.055\,kg}}$$

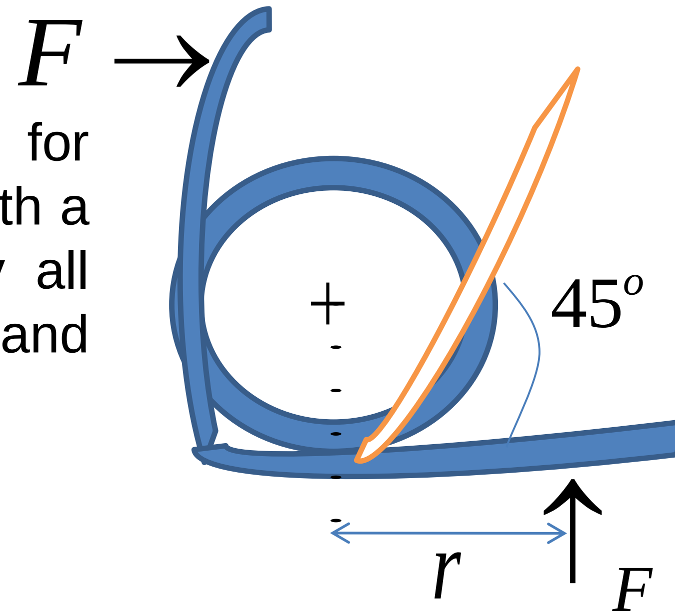
$$k = \underline{\underline{5850\ N/m}}$$

$$f_n = \frac{1}{2}\sqrt{\frac{5850}{5.055}} = 17 \quad Hz$$

$$f_f \le \frac{17}{15} \cong \quad 1Hz\,(cycle/\sec)$$

$$f_{f\,\max} = 1\ cycle/\sec$$

EXAMPLE 8: Torsion Springs



Design a straight ended helical torsion spring for static loading of 100 *Nm* at a deflection of 45° with a safety factor of 1.25 for static loading. Specify all parameters necessary to manufacture the spring and state all the assumptions.

For r = 100 mm
F=1000 N (100kg)

$$T_{\max} = (Fr)_{\max} = 100\ Nm = 100\,000\ Nmm$$

$$\theta_{\max} = 45^o = \frac{1}{8} turns(revs)$$

$$spring\ rate \quad k' = \frac{T}{\theta} = \frac{100\ Nm}{\frac{1}{8} turns} = 800 \frac{Nm}{turns}$$

if *d and D* are known then N_a can be determined.

$$Also \quad k' = \frac{d^4 E}{10.8 D N_a} \quad or \quad N_a = \frac{d^4 E}{10.8 D k'}$$

$$Since \quad n = \frac{S_y}{\sigma_{\max}} = 1.25 \quad or \quad n = \frac{0.75\left(A/d^m\right)}{K_i \frac{32(Fr)_{\max}}{\pi d^3}}$$

where

$$K_i = \frac{4C^2 - C - 1}{4C(C-1)}$$

$$K_o = \frac{4C^2 + C - 1}{4C(C+1)}$$

Use tabulation method of iteration(assume oil tempered with *A= 1880 MPa, m=0.186, d=0.5-12 mm*)

Assume d, mm	**C=8**	**D, mm**	S_y, MPa	σ_{max}, MPa	$n = \frac{S_y}{\sigma}$	**NOTES**
4		32	1089	17548	0.062	Not safe
10		80	918	1123	0.817	Not safe
11		88	902	844	1.068	Not safe
12		96	888	650	1.36	SAFE
	C=12					
10		120	918	1085.7	0.845	Not safe
11		132	902	816	1.105	Not safe
12		144	888	628.5	1.41	SAFE

$$C = \frac{D}{d}$$

for

$$C = 8; \quad K_i = 1.102$$
$$C = 4; \quad K_i = 1.23$$
$$C = 12; \quad K_i = 1.066$$

$$N_a = \frac{d^4 E}{10.8 k' D}$$

$$N_a = \frac{\left(12\times10^{-3}\right)^4 \times 207\times10^9}{10.8\times800\times0.096}$$

$$N_a = 5.175\, turns \cong 5.25\, turns$$

$$d = 12\, mm \quad D = 96\, mm$$

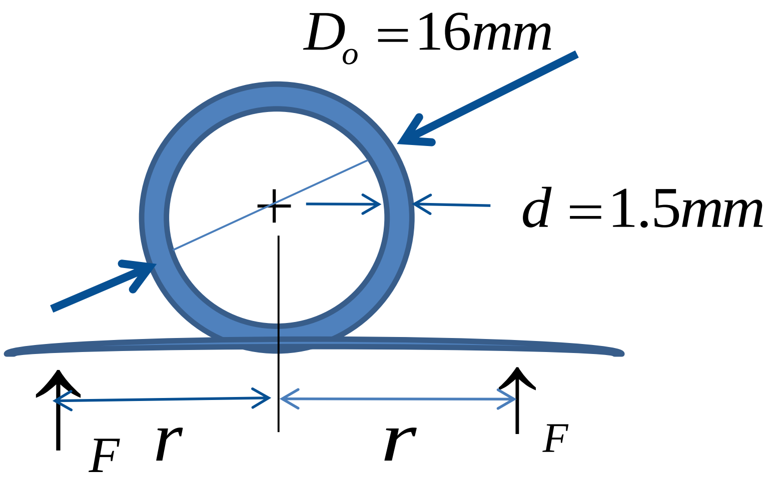
$$N_a = 5.25\, turns$$

made of oil tempered wire

EXAMPLE 9: Torsion Springs (10-20) Continue of Ex. 5)

A stock torsion spring is made of 1.5 mm music wire, has 6 coils and straight ends 50 mm long and 180º apart. The outside diameter is 16 mm.

a) What value of torque *(Fxr)* Would cause a maximum stress equal to the yield strength?
b) If the torque found in(a) is used as the maximum working torque, what is the smallest value of the inside diameter. (D_i'=?)
c) Compute the angle of rotation corresponding to the torque found in (a)



d) If the spring is to be used in an application subject to fatigue loading based on the information;

-R= 95%,
-max torque= M,
-min torque= 0.25 M,
-infinite life.

What value of the maximum torque M can safety be applied?

e) What number of cycle would be possible to run if a loading of M_{min}= 0.25 M_{max} and M_{max} = 1.0 M_{max} is applied with M_{max} =0.4 Nm? (N= ?)

SOLUTION:

a, b and c were previously solved

$$Fr_{\max} = \frac{S_y \pi d^3}{32K_i} = \frac{1534\times10^6\times\pi\left(1.5\times10^{-3}\right)^3}{32\times1.08}$$

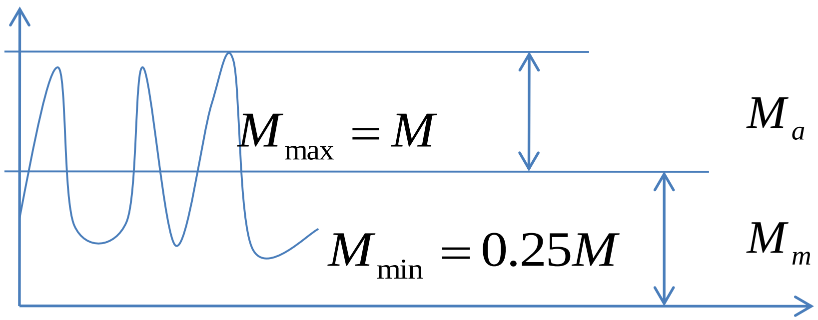
$$Fr_{\max} = 0.47\ Nm = 470\ Nmm$$

$$\Delta N = \frac{Fr}{k'}, k' = \frac{d^4E}{10.8DN_a} = \frac{\left(1.5\times10^{-3}\right)^4\times207\times10^9}{10.8\times\left(0.0145\right)\times6} = 1.115\frac{Nm}{turn}$$

$$\Delta N = n = \frac{0.47}{1.115} = 0.421 \quad turns = 151.56\ degrees$$

$$D_i' = D_i\frac{N_a}{N_a+\Delta N} = 13\frac{6}{6+0.421} = 12.147\ mm$$

d) Fatigue safety?



$$M_a = \frac{M_{max} - M_{min}}{2} = \frac{M - 0.25M}{2} = \frac{0.75M}{2} = 0.375M$$

$$M_m = \frac{M_{max} + M_{min}}{2} = \frac{1 + 0.25M}{2} = \frac{1.25M}{2} = 0.625M$$

Based on Modified Goodman theory of fatigue failure

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n} \quad \text{for infinite life}$$

Or based on Soderberg approach

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \quad \text{for infinite life}$$

In both cases

$$\sigma_a = K_i \frac{32M_a}{\pi d^3} \qquad \sigma_m = K_i \frac{32M_m}{\pi d^3}$$

$$\sigma_a = 1.08 \frac{32 \times 0.375M}{\pi d^3} \qquad \sigma_m = 1.08 \frac{32 \times 0.625M}{\pi d^3}$$

$$\sigma_a = \underline{\underline{1.222 \times 10^9 \ M}} \ (N/m^2) \qquad \sigma_m = \underline{\underline{2.037 \times 10^9 \ M}} \ (N/m^2)$$

$(\text{M in N} \cdot \text{m})$ $(\text{M in N} \cdot \text{m})$

$$S_e = k_a \times k_b \times k_c \times k_d \times S_e^{'}$$

S_e = endurance limit at the critical location of a machine part in the geometry and condition of use.

Where

$$S_e^{'} = 0.5\, S_{ut} \qquad if\;\; S_{ut} \le 1400\; MPa$$

$$700 MPa \qquad if\;\; S_{ut} \ge 1400\; MPa$$

$$\Rightarrow S_e^{'} = 700 MPa \quad \text{since} \quad S_{ut} = 2045\; MPa$$

for music wires
From Ex. 5

$$k_a = 0.63 \qquad k_b = 1.0$$

$$k_c = 0.868 \qquad k_d = 1.0$$

$$k_e = \text{is not used since } K_i \text{ is used, } \sigma \text{ is increased.}$$

$$S_e = 0.63 \times 1.0 \times 0.868 \times 1.0 \times 700$$

$$S_e = \underline{\underline{383 \quad MPa}}$$

S_{se} = endurance limit strength in shear with reliability life, temperature and stress concentration factor.

i) Based on Modified Goodman theory of fatigue failure

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n}$$

$$\frac{1.222\times10^9 M}{383\times10^6} + \frac{2.037\times10^9 M}{2045\times10^6} = \frac{1}{1}$$

$$M = \underline{\underline{0.239}} \left(N\cdot m\right)$$

ii) Based on Soderberg approach of fatigue failure

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n}$$

$$\frac{1.222\times10^9 M}{383\times10^6} + \frac{2.037\times10^9 M}{1534\times10^6} = \frac{1}{1}$$

$$M = \underline{\underline{0.221}} \left(N\cdot m\right)$$

5.13 FATIGUE LOADING

While LEWIS equation is used for static bending stress calculation AGMA equation is used for fatigue condition and gives the bending stress in tooth root under a force of W_t acting tangent to the pitch circle and including effects of stress concentration (J).

For a fatigue-free safe operation the bending stress (σ) obtained from AGMA equation should be compared with the endurance strength (S_e) of the gear material with a global safety factor n_G, that is;

$$S_e = k_a \cdot k_b \cdot k_c \cdot k_d \cdot k_e \cdot k_f \times S_e'$$

k_a: surface factor; k_b: size factor; k_c: relb. factor; k_d: temp. factor; k_e: str. con. factor; k_f: misc. factor

$$\sigma = \frac{W_t}{F \times m \times J \times K_v} \leq S_e / n_G$$

For Steels:

$$S_e' = 0.5 S_{ut} \quad \text{If } S_{ut} \leq 1400\ MPa$$

$$S_e' = 700\ MPa \quad \text{If } S_{ut} > 1400\ MPa$$

For Cast Irons:

$$S_e' = 0.45 S_{ut} \quad \text{If } S_{ut} \leq 600\ MPa$$

$$S_e' = 275\ MPa \quad \text{If } S_{ut} > 600\ MPa$$

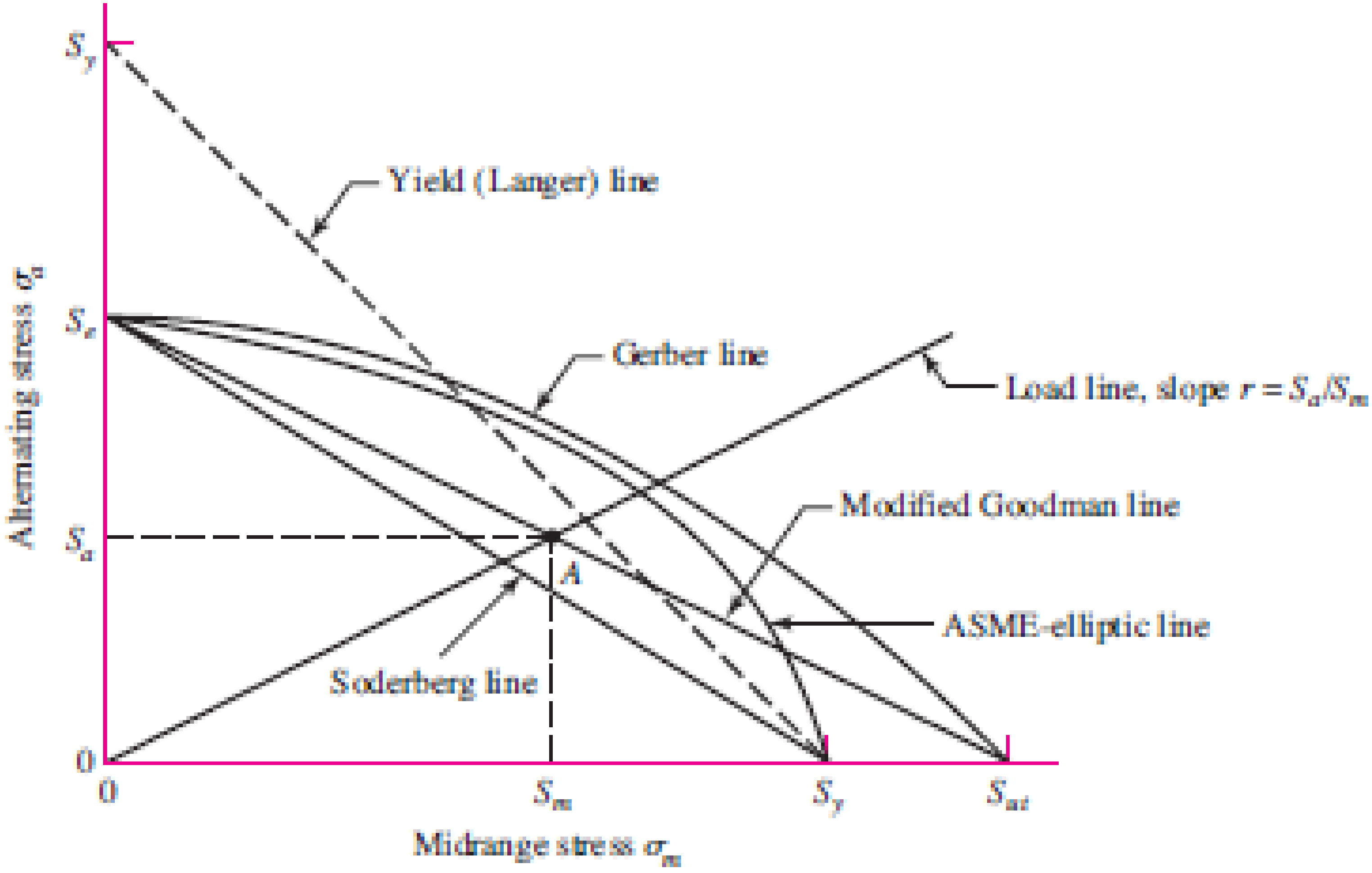


Figure 6–27 Fatigue diagram showing various criteria of failure. For each criterion, points on or "above" the respective line indicate failure. Some point *A* on the Goodman line, for example, gives the strength *Sm* as the limiting value of σ*m* corresponding to the strength *Sa*, which, paired with σ*m*, is the limiting value of σ*a*.

Surface finish: for factor k_a, use machined surface always since the tooth root is always in machined or cast form even if the tooth flank is ground.

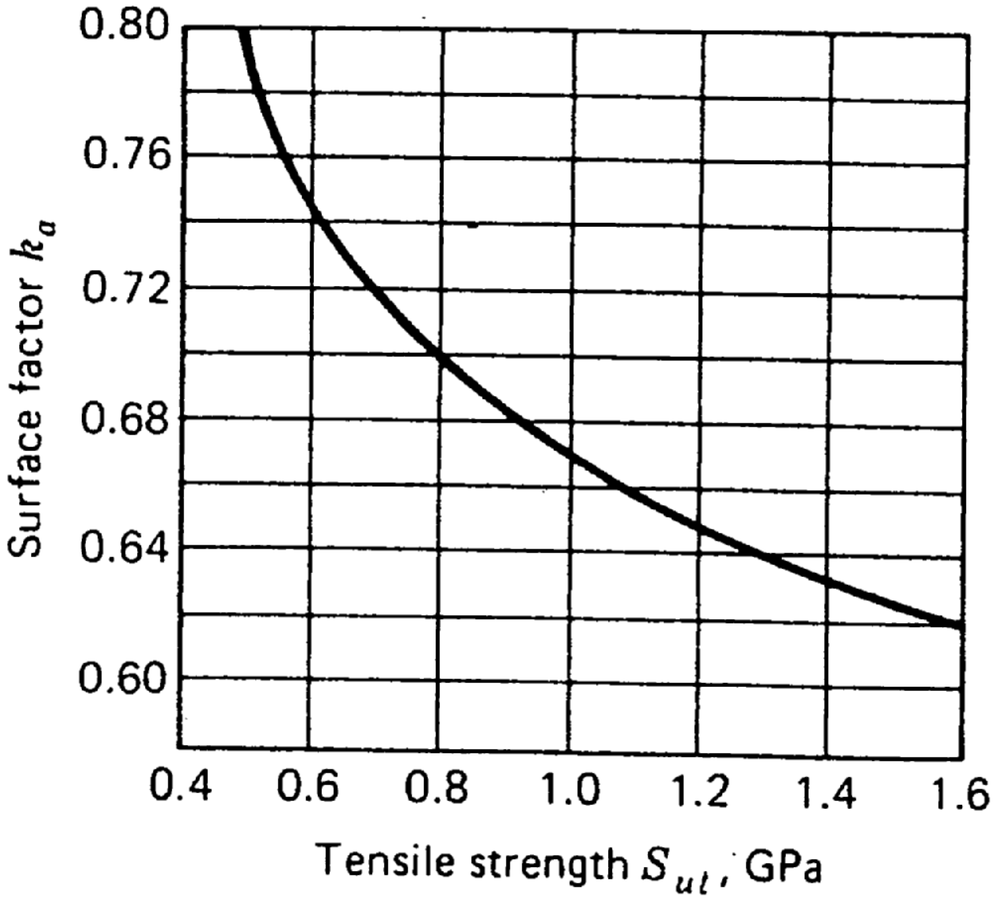


Figure 13–25 Surface finish factors k_a for cut, shaved and ground gear teeth.

<u>If the gear material is Cast Iron</u>

The S_e' values given in Table A-21 are fully corrected for surface factors (k_a), thus use k_a=1 but not corrected for other factors.

Size: The size factor, from eq. (7.16) is

$$k_b = \begin{cases} 1 & d \le 8\,mm \\ 1.189 d^{-0.097} & 8\,mm < d \le 250\,mm \end{cases} \qquad (7.16) \qquad \text{(a)}$$

For factor k_b, Eq. 7-16 is generally used but, in this equation the dimension d is the diameter of a round specimen. A spur gear tooth has a rectangular cross section and so the method of Sec. 7-7 must be used to get an equivalent value for d. For a rectangular cross section the formula for the equivalent diameter is

$$d = 0.808(hb)^{1/2} \qquad \text{(b)}$$

where *h* is the height of the section and *b* is the width. For a gear tooth *h* is the tooth thickness which is about half the circilar pitch. And *b* is the face with *F*. Substituting *h*= *p/2* and *F*= *3p* in Eq. (b) and solving gives

$$d_{eq} \cong p = \pi m \qquad \text{(c)}$$

Thus we can use these three equation to work out a set of size factors based on the module.

By using different module values h and b were determined and then k_b values were calculated. The results for k_b were then simply tabulated in Table 13-7 for different modules, and k_b values are taken from Table 13-7.

Table 13-7 SIZE FACTORS FOR SPUR-GEAR TEETH (Preferred modules in bold face

Module m	Factor k_b	Module m	Factor k_b
1 to 2	1.000	11	0.843
2.25	0.984	**12**	0.836
2.5	0.974	14	0.824
2.75	0.965	**16**	0.813
3	0.956	18	0.804
3.5	0.942	**20**	0.796
4	0.930	22	0.788
4.5	0.920	**25**	0.779
5	0.910	28	0.770
5.5	0.902	**32**	0.760
6	0.894	36	0.752
7	0.881	**40**	0.744
8	0.870	45	0.736
9	0.860	**50**	0.728
10	0.851		

THE
END