Testing the Equality of Variances (F-Distribution)

The F-distribution can be used to test the hypothesis that the variances σ_1^2 and σ_2^2 of two normally distributed populations are equal.

It is not necessary to assume that the two populations have equal means.

Population 1 σ_1^2 and Population 2 σ_2^2

F-distribution is used to test if

$\sigma_1^2 = \sigma_2^2$ or $(\sigma_1 = \sigma_2)$.

OR

Population σ^2

Sample 1 s_1^2

Sample 2 s_2^2

$\Rightarrow$ F-distribution is used if $s_1^2 = s_2^2$.

In general

$\boxed{\text{Pop. 1}}$ σ_1^2

$\boxed{\text{Pop. 2}}$ σ_2^2

Take sample 1 s_1^2

Sample 2 s_2^2

Compute the ratio of the sample variance F.

- When independent random samples have been drawn from the respective populations, the ratio

$$F = \frac{s_1^2}{s_2^2}$$ could be used to make inferences about the ratio σ_1^2 / σ_2^2.

$$F_{calc} = \frac{\text{Larger sample variance}}{\text{Smaller sample variance}}$$

V_1: The numerator degrees of freedom.

$$V_1 = n_1 - 1$$

V_2: The denominator degrees of freedom

$$V_2 = n_2 - 1$$

- If the null hypothesis (H_o) were true ($\sigma_1^2 = \sigma_2^2$), then, we would expect the ratio of the sample variances to be close to 1.0.

$$F = \frac{s_1^2}{s_2^2} \cong 1 \Rightarrow \sigma_1^2 = \sigma_2^2 : H_o$$

- If $\frac{s_1^2}{s_2^2} \cong 0$ or $\frac{s_1^2}{s_2^2} >>> 1.0 \Rightarrow$

the population variances are different ($\sigma_1^2 \neq \sigma_2^2$)

i.e., reject H_0.

Properties of F-Distribution

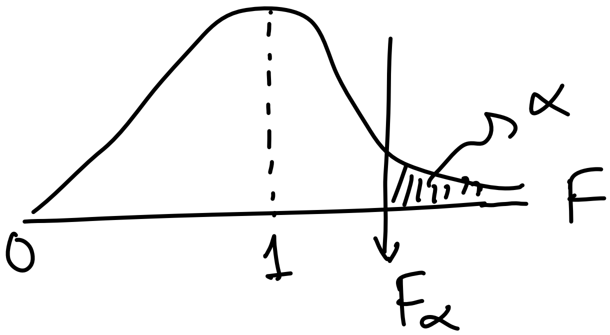
1) Unlike t or Z, F can assume only positive values.

2) It is nonsymmetrical

3) The F values are fixed from tables by α, γ_1, γ_2.

4) There are different F tables for various α values ($F_{0.10}$, $F_{0.05}$, $F_{0.025}$, $F_{0.01}$ tables)

5) Mean for F-distribution is 1.0



Confidence Interval For F = $\frac{\sigma_1^2}{\sigma_2^2}$

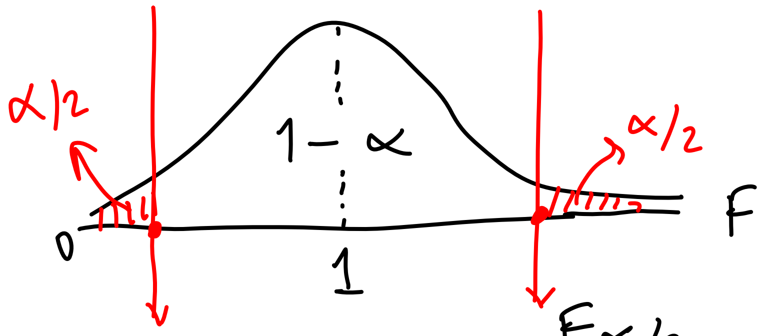
$$P\left[F_{(1-\frac{\alpha}{2})}(\nu_1,\nu_2) < F < F_{\alpha/2}(\nu_1,\nu_2)\right] = 1-\alpha$$

F: σ_1^2/σ_2^2

$F_{(1-\frac{\alpha}{2})}(\nu_1,\nu_2)$: cannot be fixed (read) from F tables.

$\Rightarrow$ Modification is needed.

$F_{\alpha/2}(\nu_1,\nu_2)$: can be fixed (read) from the F tables easily.



$F_{\alpha/2}$

$F_{(1-\frac{\alpha}{2})}$

$F_{\alpha/2}(\nu_1, \nu_2)$ = upper critical limit (point).

$F_{(1-\frac{\alpha}{2})}(\nu_1, \nu_2)$

$$\frac{1}{F_{\alpha/2}(\nu_2, \nu_1)} = \text{lower critical limit (point)}$$

ν_1: new denominator d.f.

ν_2: new numerator d.f.

$$\frac{1}{F_{\alpha/2}(\nu_2, \nu_1)} < F < F_{\alpha/2}(\nu_1, \nu_2) \ : \ CI$$

multiply both sides by $F_{\alpha/2}(\nu_2, \nu_1) \Rightarrow$

$$1 < F < F_{\alpha/2}(\nu_1, \nu_2) \times F_{\alpha/2}(\nu_2, \nu_1) \text{ and}$$

divide both sides by $F_{\alpha/2}(\nu_1, \nu_2) \Rightarrow$

$$\frac{1}{F_{\alpha/2}(V_1,V_2)} < F < F_{\alpha/2}(V_2,V_1)$$

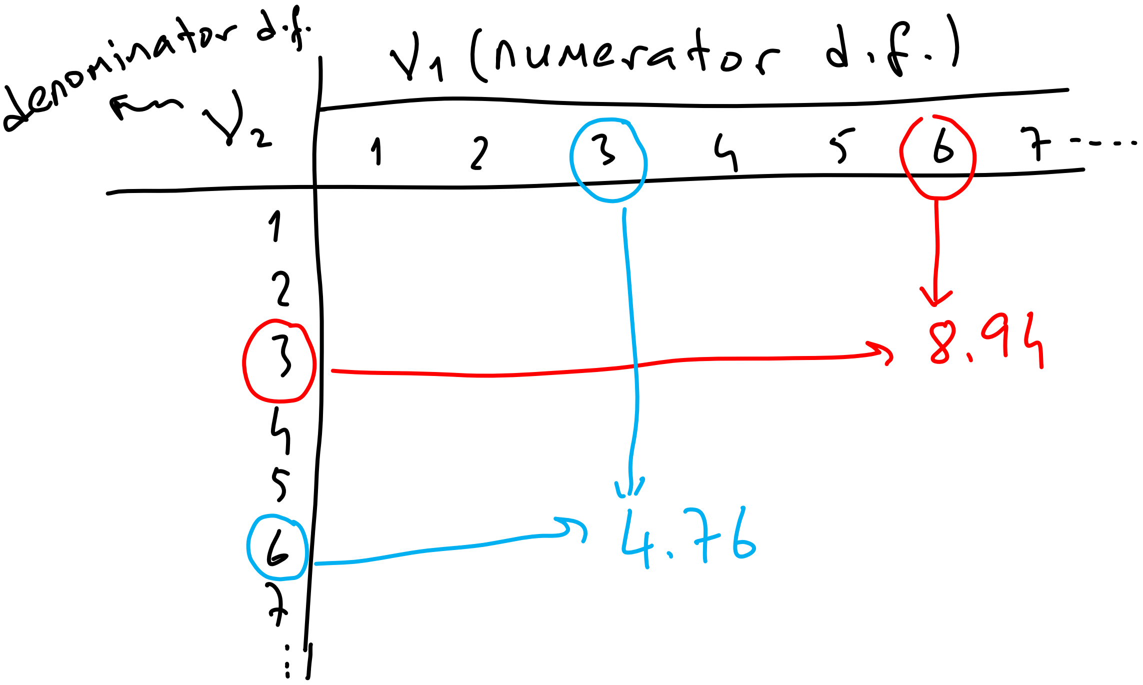
multiply both sides by $S_1^2/S_2^2 \Rightarrow$

$$CI: \boxed{\frac{S_1^2}{S_2^2} \times \frac{1}{F_{\alpha/2}(V_1,V_2)} < \underbrace{F}_{\sigma_1^2/\sigma_2^2} < \frac{S_1^2}{S_2^2} \times F_{\alpha/2}(V_2,V_1)}$$

<u>Example</u>: Determine the upper and lower critical limits for an F distribution with $V_1=6$, $V_2 = 3$ and $\alpha = 0.10$.

<u>Solution:</u> $\alpha = 0.10 \Rightarrow F_{\alpha/2} = F_{0.10/2} = F_{0.05} \Rightarrow$

Use $F_{0.05}$ table



Upper limit : $F_{\alpha/2}(V_1,V_2) = F_{0.05}(6,3) = 8.94$

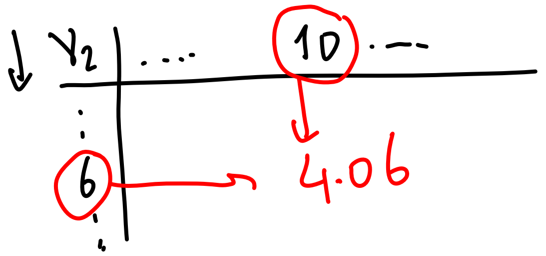
Lower limit: $\dfrac{1}{F_{\alpha/2}(\nu_2,\nu_1)} = \dfrac{1}{F_{0,05}(3,6)} = \dfrac{1}{4.76} = 0{,}21$

<u>Example</u>: What is $F_{0.95}$(6, 10) value ?

<u>Solution</u>: $F_{0,95}(6,10) \equiv \dfrac{1}{F_{0,05}(\underbrace{10}_{\text{num. d.f.}},\underbrace{6}_{\text{den. d.f.}})} = \dfrac{1}{4.06} = 0{,}246$

1.0

$F_{0.05}$ table $\Rightarrow$



<u>Example</u>: Assume we have 25 boy and 16 girl students. Average grade for the boys is 82 with a standard deviation of 8 and average grade for the girls is 78 with a standard deviation of 7. Find 98 % confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ and $\frac{\sigma_1}{\sigma_2}$.

<u>Solution</u>: $\alpha = 0{,}02,\ n_1 = 25,\ n_2 = 16,\ S_1 = 8,\ S_2 = 7$

$F_{\alpha/2} = F_{0{,}02/2} = F_{0{,}01} \Rightarrow$ Use $F_{0.01}$ table.

$\nu_1 = n_1 - 1 = 25 - 1 = 24$

$\nu_2 = n_2 - 1 = 16 - 1 = 15$

$F_{\alpha/2}(\nu_1,\nu_2) = F_{0{,}01}(24,15) = 3.29$

$F_{\alpha/2}(\nu_2,\nu_1) = F_{0{,}01}(15,24) = 2.89$

98% CI =>

$$\frac{64}{49} \cdot \left(\frac{1}{3.29}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \frac{64}{49} \times (2.89)$$

$$0.397 < \frac{\sigma_1^2}{\sigma_2^2} < 3.775 \rightsquigarrow \text{CI for } \frac{\sigma_1^2}{\sigma_2^2}$$

take $\sqrt{}$ of both sides =>

$$0.630 < \frac{\sigma_1}{\sigma_2} < 1.943 \longrightarrow \text{CI for } \frac{\sigma_1}{\sigma_2}$$

Hypothesis Testing Using F-Distribution

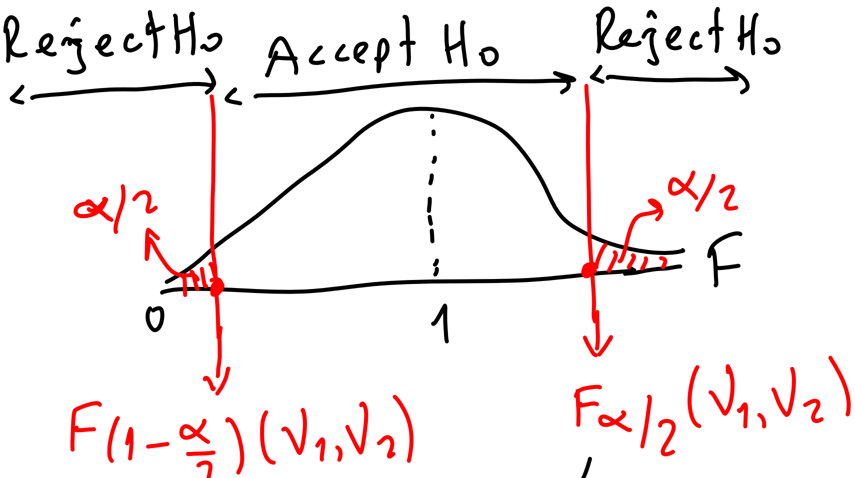
a) Two-Tailed Test

$$H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1 \quad \text{OR} \quad \sigma_1^2 = \sigma_2^2$$

$$H_A: \frac{\sigma_1^2}{\sigma_2^2} \neq 1 \quad \text{OR} \quad \sigma_1^2 \neq \sigma_2^2$$

Test statistic : F_{calc}

$$F_{calc} = \frac{\text{Larger Variance}}{\text{Smaller Variance}}$$



$F_{calc} = \dfrac{s_1^2}{s_2^2}$ when $s_1^2 > s_2^2$

OR

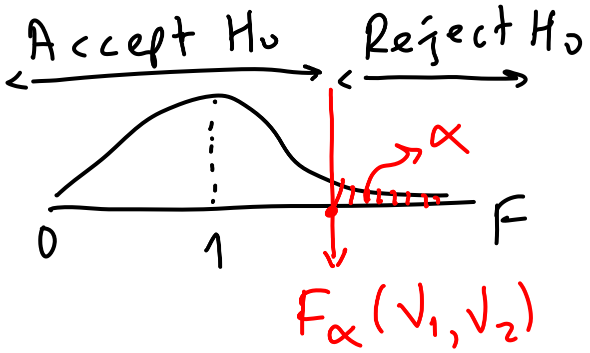
$F_{calc} = \dfrac{s_2^2}{s_1^2}$ when $s_2^2 > s_1^2$

Critical points
(Table values)

b) One-Tailed Test (Right Hand Sided)

$H_0: \dfrac{\sigma_1^2}{\sigma_2^2} = 1$ or $\sigma_1^2 = \sigma_2^2$

$H_A: \dfrac{\sigma_1^2}{\sigma_2^2} > 1$ or $\sigma_1^2 > \sigma_2^2$



Critical point
(Table value)

Test statistic value: F_{calc}

$$F_{calc} = \frac{\text{Larger Variance}}{\text{Smaller Variance}}$$

$$F_{calc} = \frac{s_1^2}{s_2^2} \quad \text{when } s_1^2 > s_2^2 \quad \text{OR}$$

$$F_{calc} = \frac{s_2^2}{s_1^2} \quad \text{when } s_2^2 > s_1^2$$

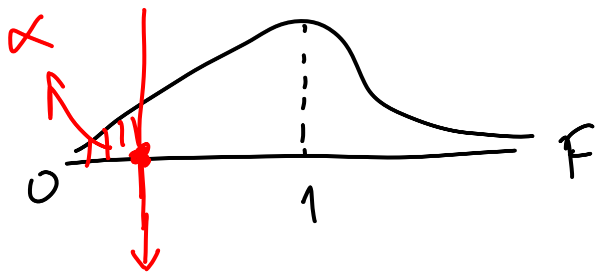
c) One-Tailed Test (Left Hand Sided)

$$H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1 \quad \text{or} \quad \sigma_1^2 = \sigma_2^2$$

$$H_A: \frac{\sigma_1^2}{\sigma_2^2} < 1 \quad \text{or} \quad \sigma_1^2 < \sigma_2^2$$

Test statistic value: F_{calc}

$$F_{calc} = \frac{\text{Smaller Variance}}{\text{Larger Variance}}$$



$F_{(1-\alpha)}(\nu_1, \nu_2)$

$\Downarrow$

Critical point

$\Downarrow$

$$F_{(1-\alpha)}(\nu_1, \nu_2) \equiv \frac{1}{F_{\alpha}(\nu_2, \nu_1)}$$

<u>Example</u>: If $s_1^2 = 48.7$, $n_1 = 5$, $s_2^2 = 3.7$ and $n_2 = 6$, then, test whether the two populations have the same variabilities at $\alpha = 0.10$.

<u>Solution</u>: Two - tailed test.

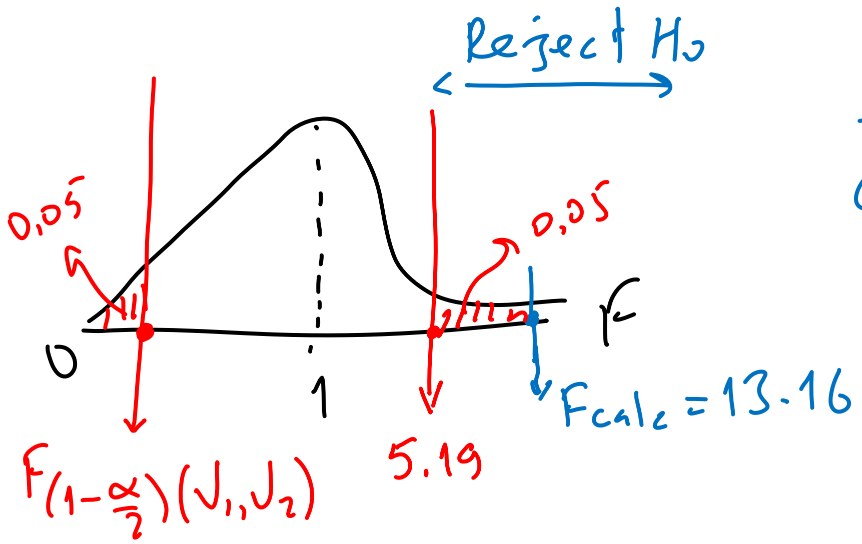
X $H_0: \sigma_1^2 = \sigma_2^2$ OR $s_1^2 = s_2^2$

✓ $H_A: \sigma_1^2 \neq \sigma_2^2$ $s_1^2 \neq s_2^2$

$$F_{calc} = \frac{\text{Larger var.}}{\text{Smaller var.}} = \frac{48.7}{3.7} = 13.16$$

Table values: $\nu_1 = 5 - 1 = 4$, $\nu_2 = 6 - 1 = 5$

$$F_{\alpha/2}(\nu_1, \nu_2) = F_{0,05}(4, 5) = 5.19$$



Decision: Reject Ho

Conclusion: The two populations have different variabilities.